Problem 1. Let \( \phi : M_{\lambda} \to M_{\mu} \) be a nonzero homomorphism of Verma modules over a \( \mathbb{Z} \)-graded Lie algebra \( g \) (with abelian \( g_0 \)). Show that \( \phi \) is injective.

Problem 2. A quantum field is a formal series \( A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \), where \( A_n \) are operators in some vector space, or more generally elements of some noncommutative algebra. Let \( A_+(z) = \sum_{n < 0} A_n z^{-n-1} \) be the Taylor part, and \( A_-(z) = A(z) - A_+(z) \). For two quantum fields \( A, B \), the normal ordered product \( : A(z)B(w) : \) is defined by the formula

\[
: A(z)B(w) := A_+(z)B(w) + B(w)A_-(z).
\]

Consider the quantum fields \( a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \) for the Heisenberg algebra \( A \) and \( T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) for the Virasoro algebra \( Vir \).

(i) Compute the difference \( a(z)a(w) - : a(z)a(w) : \) on the Fock representation \( F_\mu \) (sum the power series you get and express as a rational function in \( z, w \). In fact, this function should depend only on \( z - w \)).

(ii) Regard the Fock representation \( F_\mu \) as a module over \( Vir \ltimes A \). In this module, compute \( T(z)a(w) - : T(z)a(w) : \) as a linear combination of \( a(w) \) and its derivatives with coefficients being rational functions of \( z, w \) (in fact, of \( z - w \)).

(iii) Let \( V \) be a highest weight Virasoro module with central charge \( c \). Compute \( T(z)T(w) - : T(z)T(w) : \) on \( V \) as a linear combination of \( T(w) \) and its derivatives with coefficients being rational functions of \( z, w \) plus a rational function of \( z, w \) (in fact, these will be functions of \( z - w \)).