18.747: Problem set 3; due Thursday, September 26

Problem 1. Let $\phi : M_\lambda \to M_\mu$ be a nonzero homomorphism of Verma modules over a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ (with abelian $\mathfrak{g}_0$). Show that ϕ is injective.

Problem 2. A quantum field is a formal series $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$, where A_n are operators in some vector space, or more generally elements of some noncommutative algebra. Let $A_+(z) = \sum_{n<0} A_n z^{-n-1}$ be the Taylor part, and $A_-(z) = A(z) - A_+(z)$. For two quantum fields A, B, the normal ordered product $: A(z)B(w) :$ is defined by the formula

$$: A(z)B(w) := A_+(z)B(w) + B(w)A_-(z).$$

Consider the quantum fields $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ for the Heisenberg algebra $\mathcal{A}$ and $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ for the Virasoro algebra Vir.

(i) Compute the difference $a(z)a(w)- : a(z)a(w) :$ on the Fock representatiion F_μ (sum the power series you get and express as a rational function in z, w. In fact, this function should depend only on $z - w$).

(ii) Regard the Fock representation F_μ as a module over Vir $\ltimes \mathcal{A}$. In this mocule, compute $T(z)a(w)- : T(z)a(w) :$ as a linear combination of $a(w)$ and its derivatives with coefficients being rational functions of z, w (in fact, of $z - w$).

(iii) Let V be a highest weight Virasoro module with central charge c. Compute $T(z)T(w)- : T(z)T(w) :$ on V as a linear combination of $T(w)$ and its derivatives with coefficients being rational functions of z, w plus a rational function of z, w (in fact, these will be functions of $z - w$).