18.747: Problem set 2; due Thursday, September 19

1. In the Fock representation F_μ of the Heisenberg algebra, define the operators

$$L_k = \frac{1}{2}\sum_{j\in Z} a_{-j}a_{j+k} + i\lambda k a_k, k \neq 0,$$

,

$$L_0 = \frac{\lambda^2+\mu^2}{2} + \sum_{j>0} a_{-j}a_j.$$

Prove that these operators define a representation of Vir on F_μ with central charge $c = 1 + 12\lambda^2$.

2. Prove Proposition 3.7 in Kac-Raina.

3. In the Heisenberg algebra $\mathcal{A}$, define "the quantum field" $a(z) = \sum_j a_j z^{-j-1}$. Let F_0 be the Fock representation of $\mathcal{A}$ (with $a_0 = 0$), and let 1 denote its highest weight vector. Let 1^* be the lowest weight vector of the dual representation.

(i) Prove that

$$< 1^*, a(z)a(w)1 >= \frac{1}{(z-w)^2}$$

where the right hand side means the Taylor expansion of $1/(z-w)^2$ as a function of w in the region $|w| < |z|$ (for $z \neq 0$).

(ii) Show that

$$< 1^*, a(z_1)....a(z_{2n})1 >= \sum_{\{s\in S_{2n}; s^2=1, s(i)\neq i \forall i\}} \prod_{i<s(i)} \frac{1}{(z_i - z_{s(i)})^2},$$

where the right hand side means the expansion in the region $|z_1| > ... > |z_{2n}|$.