1. In the Fock representation $F_\mu$ of the Heisenberg algebra, define the operators

$$L_k = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j}a_{j+k} + i\lambda ka_k, k \neq 0,$$

$$L_0 = \frac{\lambda^2 + \mu^2}{2} + \sum_{j>0} a_{-j}a_j.$$

Prove that these operators define a representation of Vir on $F_\mu,1$ with central charge $c = 1 + 12\lambda^2$.

2. Prove Proposition 3.7 in Kac-Raina.

3. In the Heisenberg algebra $A$, define “the quantum field” $a(z) = \sum_j a_jz^{-j-1}$.

Let $F_0$ be the Fock representation of $A$ (with $a_0 = 0$), and let $1$ denote its highest weight vector. Let $1^*$ be the lowest weight vector of the dual representation.

(i) Prove that

$$<1^*, a(z)a(w)1> = \frac{1}{(z-w)^2}$$

where the right hand side means the Taylor expansion of $1/(z-w)^2$ as a function of $w$ in the region $|w| < |z|$ (for $z \neq 0$).

(ii) Show that

$$<1^*, a(z_1)\ldots a(z_{2n})1> = \sum_{\{s \in S_{2n}, s^2 = 1, s(i) \neq i \forall i\}} \prod_{1 < s(i)} \frac{1}{(z_i - z_{s(i)})^2},$$

where the right hand side means the expansion in the region $|z_1| > \ldots > |z_{2n}|$. 