18.747: Problem set 12; due Thursday, December 5

1. Let $M = M_{\lambda,k}$ be the Verma module over $\widehat{sl}_2$, of level $k \neq -2$, and V the irreducible representation of sl_2 of dimension $2m+1$. Regard V as a representation of $\widehat{sl_2}$ by $(at^n)v = av$, $a \in sl_2$, and $Kv = 0$.

(a) Show that for generic λ, k there exists a unique, up to scaling, intertwining operator $\Phi : M \to M\hat{\otimes}V$ (here $\hat{\otimes}$ is the completed tensor product).

(b) Define the function of two variables

$$F(x,t) := \mathrm{Tr}_M(\Phi e^{tL_0+xh}),$$

where $h \in sl_2$ is the Cartan element, and L_0 is given by the Sugawara construction. This function takes values in $V[0]$, the zero weight subspace of V, which is 1-dimensional, so we may regard it as a scalar function. Show that $F(x,t)$ satisfies a linear parabolic PDE of the form

$$F_t = \gamma F_{xx} + \text{lower terms},$$

whose coefficients are elliptic functions. Compute the exact form of this equation.