18.747: Problem set 12; due Thursday, May 15

1. Let $M = M_{\lambda,k}$ be the Verma module over $\hat{sl}_2$, of level $k \neq -2$, and $V$ the irreducible representation of $sl_2$ of dimension $2m + 1$. Regard $V$ as a representation of $\hat{sl}_2$ by $(at^n)v = av$, $a \in sl_2$, and $Kv = 0$.

(a) Show that for generic $\lambda, k$ there exists a unique, up to scaling, intertwining operator $\Phi : M \rightarrow M \hat{\otimes} V$ (here $\hat{\otimes}$ is the completed tensor product).

(b) Define the function of two variables $F(x,t) := \text{Tr}_M(\Phi e^{tL_0 + xh})$, where $h \in sl_2$ is the Cartan element, and $L_0$ is given by the Sugawara construction. This function takes values in $V[0]$, the zero weight subspace of $V$, which is 1-dimensional, so we may regard it as a scalar function. Show that $F(x,t)$ satisfies a linear parabolic PDE of the form

$$F_t = \gamma F_{xx} + \text{lower terms},$$

whose coefficients are elliptic functions.