18.747: Problem set 11; due Thursday, November 21

1. For a complex n by n matrix, let $\mathbf{g}(A)$ be the corresponding contragredient Lie algebra.
(a) Suppose that A is a generalized Cartan matrix, and assume that for some $m \geq 0$, one has $a_{ij} \leq -m$ for all $i \neq j$. Use the Weyl-Kac denominator formula to show that the subalgebra $\mathbf{n}_+ \subset \mathbf{g}(A)$ is free in degrees $1 \leq d \leq m+1$ (i.e., there is no relations between e_i in these degrees, and similarly for f_i). This is a fairly weak form of the Gabber-Kac theorem, stating that the Serre relations are the defining relations among the e_i (and f_i).
Hint. The inverse of the Weyl-Kac denominator is the character of $U(\mathbf{n}_+)$, and the problem is to show that this is a free associative algebra in degrees $\leq m+1$.
(b) Deduce that for a Weil generic complex matrix A (outside of a countable set of hypersurfaces), the elements e_i of $\mathbf{g}(A)$ generate a free Lie algebra, and so do the elements f_i. (You do not have to find the hypersurfaces explicitly).
2. Let $L_{\lambda,k}$ be an integrable highest weight module over the affine Kac-Moody Lie algebra $\widehat{\mathbf{g}}$ with highest weight λ and level k. Let V be a finite dimensional $\widehat{\mathbf{g}}$-module. Show that the space $\mathrm{Hom}_{\widehat{\mathbf{g}}}(L_{\lambda,k} \otimes L^*_{\mu,k}, V)$ is isomorphic to the space of vectors $v \in V[\lambda - \mu]$ such that $f_i^{\lambda(h_i)+1} v = 0$ for all $i = 0, ..., r$.