18.747: Problem set 10; due Thursday, April 24

1. Show that every integrable module M of finite length in category O over a Kac-Moody algebra is a direct sum of irreducible modules.
Hint. It suffices to show that for every irreducible integrable modules V, W from O, one has $\mathrm{Ext}^1(V, W) = 0$ (where Ext is taken in category O). To show this, argue that for an exact sequence $0 \to L(a) \to X \to L(b) \to 0$ to be non-split, it is necessary that $a - b$ be a nonnegative integer combination of simple roots. Deduce from this that for $a \neq b$ the eigenvalue of the Casimir on $L(a)$ is larger than that for $L(b)$, and deduce triviality of the extension. The case $a = b$ should be considered separately.
2. Using the Casimir, show that for generic λ, the Verma module $M(\lambda)$ over a Kac-Moody algebra is irreducible.