18.747: Problem set 10; due Thursday, November 14

1. Show that every integrable module M of finite length in category O over a Kac-Moody algebra is a direct sum of irreducible modules.
Hint. It suffices to show that for every irreducible integrable modules V, W from O, one has $\mathrm{Ext}^1(V, W) = 0$ (where Ext is taken in category O). To show this, argue that for an exact sequence $0 \to L_a \to X \to L_b \to 0$ to be non-split, it is necessary that $a - b$ be a nonnegative integer combination of simple roots. Deduce from this that for $a \neq b$ the eigenvalue of the Casimir on L_a is larger than that for L_b, and deduce triviality of the extension. The case $a = b$ should be considered separately.
2. (a) Using the Casimir, show that for generic λ, the Verma module M_λ over an extended Kac-Moody algebra is irreducible. Specify a countable collection of hyperplanes outside of which it is true.
(b) Let N be a module over an extended Kac-Moody algebra from category O. Show that for a Weil generic λ (more precisely, for λ outside of a countable collection of hyperplanes), the module $M_\lambda \otimes N$ is semisimple, and describe its decomposition into irreducibles.