18.747: Problem set 1; due Thursday, September 12

1. Let $\alpha, \beta \in \mathbf{C}$ and $V_{\alpha,\beta}$ be the space of formal expressions $g(z)z^\alpha(dz)^\beta$, where g is a Laurent polynomial ("tensor fields" of rank β and branching α on the punctured complex plane).
(a) Show that the formula

$$f\partial \circ gz^\alpha(dz)^\beta = (fg' + \alpha z^{-1}fg + \beta f'g)z^\alpha(dz)^\beta$$

defines an action of W on $V_{\alpha,\beta}$.
(b) Recall that the Lie algebra W has a basis $L_n, n \in \mathbf{Z}$ with $[L_n, L_m] = (n-m)L_{m+n}$. Let $v_k = z^{k+\alpha}(dz)^\beta$ (this is a basis of $V_{\alpha,\beta}$). Check that

$$L_n \circ v_k = -(k + \alpha + (n+1)\beta)v_{n+k}.$$

Deduce that this formula defines a representation of W.

2. Find the necessary an sufficient conditions on $(\alpha, \beta, \alpha', \beta')$ under which $V_{\alpha,\beta}$ is isomorphic to $V_{\alpha',\beta'}$. When is $V_{\alpha,\beta}$ irreducible?

3. Show that W is a simple Lie algebra (i.e. it has no two-sided ideals other than $0, W$). Deduce from this that W has no nontrivial finite dimensional modules (i.e. the action of W in any such module is zero).

4. Check that the formula $\omega(L_n, L_m) = n^3\delta_{n,-m}$ defines a 2-cocycle on W.

5. Show that the Virasoro algebra Vir is perfect (coincides with its commutator subalgebra). Derive that Vir is not isomorphic to $W \oplus \mathbf{C}$ as a Lie algebra.