

18.747: Problem set 1; due Thursday, Feb. 14

1. Recall that the Lie algebra W has a basis $L_n, n \in \mathbb{Z}$ with $[L_n, L_m] = (n - m)L_{m+n}$. In class we defined the W -modules $V_{\alpha, \beta}$, with basis $v_k, k \in \mathbb{Z}$, and the action of W given by $L_n v_k = -(k + \alpha + (n + 1)\beta)v_{n+k}$. Check by a direct calculation that this formula indeed defines a representation.
2. Find the necessary and sufficient conditions on $(\alpha, \beta, \alpha', \beta')$ under which $V_{\alpha, \beta}$ is isomorphic to $V_{\alpha', \beta'}$. When is $V_{\alpha, \beta}$ irreducible?
3. Show that W is a simple Lie algebra (i.e. it has no two-sided ideals other than $0, W$). Deduce from this that W has no nontrivial finite dimensional modules (i.e. the action of W in any such module is zero).
4. Check that the formula $\omega(L_n, L_m) = n^3 \delta_{n, -m}$ defines a 2-cocycle on W .
5. Show that the Virasoro algebra Vir is perfect (coincides with its commutator subalgebra). Derive that Vir is not isomorphic to $W \oplus \mathbb{C}$ as a Lie algebra.