Lectures 8-9

Let us now compute the character of $\Lambda^{\frac{\infty}{2},m}V$. We define

$$\bar{d} = \deg(v_{i_0} \wedge v_{i_1} \wedge \cdots) = -\sum_{k\geq 0}(i_k + k - m) \leq 0.$$

Note that $i_k + k - m$ is a decreasing sequence which is eventually 0, so it defines a partition of d.
Conversely, any partition λ defines a vector by $i_k = m - k + \lambda_k$.
Thus,

$$\underset{\text{ch}\,\Lambda^{\frac{\infty}{2},m}V(q)}{\underset{\parallel}{\sum \dim \Lambda^{\frac{\infty}{2},m}V[-d]\, q^d}} = \sum_{d\geq 0} p(d) q^d = \frac{1}{(1-q)(1-q^2)\cdots}$$

<u>Corollary</u> $\Lambda^{\frac{\infty}{2},m}V \cong F_m$ as a representation of $\mathcal{A}$

<u>Proof</u> Let $\psi_m = v_m \wedge v_{m-1} \wedge \cdots$

Then $T^i \psi_m = 0$ for $i > 0$.

Also $T^0 \psi_m = \mathbb{I} \cdot \psi_m = \hat{\rho}(\mathbb{I})\psi_m = \underbrace{(1 + \cdots + 1)}_{m} \psi_m = m\psi_m$

So we have a nonzero grading preserving map $\sigma : F_m \to \Lambda^{\frac{\infty}{2},m}V$, $\sigma(\mathbb{1}) = \psi_m$. As F_m is irreducible,

this map is injective. Hence σ is also surjective, as the degrees of the homogeneous components on both sides are the same.
So σ is an isomorphism.
So we now have a new realization of the Fock module F_m by semiinfinite wedges. We want to compare the two realizations.
I will denote $\Lambda^{\frac{\infty}{2},m}V$ by $\mathcal{F}^{(m)}$, F_m by $\mathcal{B}^{(m)}$, and set $\mathcal{F}=\bigoplus_m \mathcal{F}^{(m)}$, $\mathcal{B}=\bigoplus_m \mathcal{B}^{(m)}$. So we have an isomorphism $\sigma=\bigoplus_m \sigma_m : \mathcal{F}\to\mathcal{B}$ preserving the decomposition. This is called the Boson-Fermion correspondence. (σ is inverse to the natural map $F_m \to \mathcal{F}^{(m)}$). To distinguish $\mathcal{B}^{(m)}$ in $\mathcal{B}$, introduce a new variable z, and set
$\mathcal{B}^{(m)}=z^m\mathbb{C}[x_1,x_2,\ldots]$. So $\mathcal{B}=\mathbb{C}[z,z^{-1},x_1,x_2,\ldots]$.
We have the following questions about comparison of the two realizations:
1) What polynomials arise as images of monomials in $\mathcal{F}$

2) How to extend $\mathcal{B}$ from $\mathcal{A}$-module to $\mathfrak{a}_\infty$-module explicitly.

We will start with the second question. First we introduce Fermionic operators on F. For each i, let $\hat{v}_i : F \to F$ be the wedging operator with v_i, and $\check{v}_i : F \to F$ the contraction operator with v_i.

ξ_i " $\hat{v}_i$, ξ_i^* " $\check{v}_i$

It is easy to see that

$$\hat{v}_i : F^{(m)} \to F^{(m+1)}, \quad \check{v}_i : F^{(m)} \to F^{(m-1)},$$

and $\quad \check{v}_i \hat{v}_j + \hat{v}_j \check{v}_i = \delta_{ij}$, i.e.

$$\xi_i \xi_j^* + \xi_j^* \xi_i = \delta_{ij},$$

while $\xi_i \xi_j + \xi_j \xi_i = 0, \quad \xi_i^* \xi_j^* + \xi_j^* \xi_i^* = 0.$

Now, $\quad \rho(E_{ij}) = \xi_i \xi_j^*,$

$$\hat{\rho}(E_{ij}) = \begin{cases} \xi_i \xi_j^* - 1 & \text{if } i = j \le 0 \\ \xi_i \xi_j^* & \text{otherwise.} \end{cases} =: \xi_i \xi_j^* :$$

Thus $\quad a_k = \sum : \xi_i \xi_{i+k}^* :$, so

if we set $\xi(z) = \sum \xi_n z^{-n-\frac{1}{2}}$, $\xi^*(z) = \sum \xi_n^* z^{-n-\frac{1}{2}}$,

then $\quad a(z) = : \xi(z) \xi^*(z) :$

We'd like to express $\hat{\rho}(E_{ij})$ in terms of a_i. For this it's enough to express $\hat{\xi}_i, \hat{\xi}_j^*$ in terms of a_i: This is accomplished by the Vertex operator construction.

Vertex operator construction

Set $X(u) = \sum \xi_n u^n$, $X^*(u) = \sum \xi_n^* u^{-n}$.

Recall that we have $\sigma : \mathcal{F} \to \mathcal{B}$.

Let $\Gamma(u) = \sigma X(u) \sigma^{-1}$

$\Gamma^*(u) = \sigma X^*(u) \sigma^{-1}$

Theorem. The operators $\Gamma(u) : B^{(m)} \to B^{(m+1)}$, $\Gamma^*(u) : B^{(m)} \to B^{(m-1)}$ are

$$\Gamma(u) = u^{m+1} z\, e^{\sum_{j>0} \frac{a_{-j}}{j} u^j} e^{-\sum_{j>0} \frac{a_j}{j} u^{-j}}$$

$$\Gamma^*(u) = u^{-m} z^{-1} e^{-\sum_{j=1}^{\infty} \frac{a_{-j}}{j} u^j} e^{\sum_{j=1}^{\infty} \frac{a_j}{j} u^{-j}}$$

(These are morally $:e^{\pm \int a(u)\,du}:$)

Proof

Lemma. $[a_j, \Gamma(u)] = u^j \Gamma(u)$

Proof We need $[T^j, X(u)] = u^j X(u)$

But $T^j = a_j = \sum :\xi_i \xi_{i+j}^*:$

So we need

$$[:\sum \xi_i \xi_{i+j}:, \xi_k] = \xi_{k+j},$$

which is easy. $\square$

Now let's prove the formula for $\Gamma(u)$ (the proof for $\Gamma^*(u)$ is similar).

Set

$$\Gamma_+(u) = e^{-\sum_{i>0} \frac{a_{i}}{i} u^{-i}}$$

Clearly, $[a_j, \Gamma_+(u)] = 0,\ j \geq 0$.

$$[a_j, \Gamma_+(u)] = u^j \Gamma_+(u),\ j < 0$$

(this is proved by a direct computation) using interpretation of a_j as $\frac{\partial}{\partial x_j}$ and $j x_j$)

So $[a_j, \Gamma(u)\Gamma_+(u)^{-1} z^{-1}] = \begin{cases} 0, & j \leq 0 \\ u^j \Gamma(u)\Gamma_+(u)^{-1} z^{-1}, & j > 0. \end{cases}$

(we use $[a_j, z] = 0$, $[a_0, z] = z$).

So the operator

$$\Delta(u) = \Gamma(u)\,\Gamma_+(u)^{-1} z^{-1}$$ has the following property:

$$P(a_{-1}, a_{-2}, \dots)\,\Delta(u) v_m = \Delta(u) P(a_{-1}, a_{-2}, \dots) v_m$$

Thus to know the action of $\Delta(u)$, we just need to know $\Delta(u) v_m$

(v_m is the highest weight vector of $F_m = \mathcal{F}^{(m)}$).

We have

$$a_j \cdot \Delta(u) v_m = u^j \Delta(u) v_m \quad , \quad j > 0.$$

also $\Delta(u) v_m = Q(u, x_1, x_2, \dots)$

Recall that $a_j = \dfrac{\partial}{\partial x_j}$.

So we have

$$\frac{\partial Q}{\partial x_j} = u^j Q$$

So $Q = f(u) e^{\sum_{j>0} u^j x_j}$

where f is a laurent series in u.

Thus $\Delta(u) = f(u) e^{\sum_{j>0} \frac{a_{-j}}{j} u^j}$

So $\Gamma(u) = z\, f(u) e^{\sum_{j>0} \frac{a_{-j}}{j} u^j} e^{-\sum_{j>0} \frac{a_j}{j} u^{-j}}$

So to conclude the proof, we need

to show that $\overset{-73}{f}(u) = u^{m+1}$!

To do this, just compute a matrix element:

$$\langle \psi_{m+1}^*, X(u)\psi_m \rangle = \langle \psi_{m+1}^*, \sum \xi_j u^j \psi_m \rangle$$

$= u^{m+1}$ (the only coefficient that contributes is $\xi_{m+1} = \hat{v}_{m+1}$); while

$$\langle v_{m+1}^*, \Gamma(u) v_m \rangle = f(u). \quad \blacksquare$$

<u>Corollary</u>

$$\rho\Big(\sum_{i,j} u^i v^{-j} E_{ij}\Big) = \frac{(u/v)^m}{1-\frac{v}{u}} \Gamma(u,v),$$

where

$$\Gamma(u,v) = e^{\sum_{j\geq 1} \frac{u^j - v^j}{j} a_{-j}} \, e^{-\sum_{j\geq 1} \frac{u^{-j}-v^{-j}}{j} a_j}.$$

<u>Pf</u> $\rho(\sum u^i v^{-j} E_{ij}) = X(u) X^*(v)$ since $E_{ij} = \xi_i \xi_j^*$, so the result follows ~~by~~ direct calculation.

<u>Cor</u>

$$\hat{\rho}\Big(\sum_{i,j} u^i v^{-j} E_{ij}\Big) = \frac{1}{1-\frac{v}{u}}\left(\left(\frac{u}{v}\right)^m \Gamma(u,v) - 1\right).$$

<u>Pf</u> Exercise.

Now we want to answer the question What is the image of wedge monomials under σ?

To express the answer, let us introduce Schur's polynomials.

$S_k(x) \in \mathbb{Q}[x_1, x_2, \dots]$ is defined by

$$\sum S_k(x) z^k = e^{\sum_{j=0} x_i z^i}$$

Ex. $S_0 = 1$, $S_1 = x_1$, $S_2 = \frac{1}{2} x_1^2 + x_2$

Recall also complete symmetric functions

$$h_k(y) = \sum_{\substack{p_1, \dots, p_N \\ p_1 + \dots + p_N = k}} y_1^{p_1} \cdots y_N^{p_N}$$

Ex. $h_3(y_1, y_2) = y_1^3 + y_1^2 y_2 + y_1 y_2^2 + y_2^3$

Prop. $$\sum z^k h_k(y) = \prod_{j=1}^{N} \frac{1}{1 - z y_j}$$

Pf. On both sides every monomial of degree k occurs exactly once with coeff. z^k.

Prop. (Relation between Schur and complete symmetric functions)

Prop.
If $$x_j = \frac{y_1^j + \cdots + y_N^j}{j}$$ then
$$h_k(y) = S_k(x)$$

Pf. $$\sum S_k(x) z^k = \exp\left(\sum x_i z^i\right)$$
$$= \exp\left(\sum \frac{(y_1^i + \cdots + y_N^i) z^i}{i}\right) =$$
$$\sum h_k(y) z^k$$
$$= \exp\left(-\sum \log(1 - z y_j)\right)$$
$$= \prod \frac{1}{1 - z y_j}$$

Now consider a partition $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_m)$

We define the Schur polynomials
$$S_\lambda(x) = \det \begin{pmatrix} S_{\lambda_1}(x) & S_{\lambda_1+1}(x) & \cdots & S_{\lambda_1+m-1}(x) \\ \vdots & \ddots & & \vdots \\ S_{\lambda_m-m+1}(x) & \cdots & & S_{\lambda_m}(x) \end{pmatrix}$$

(we put zero if the subscript is negative)

Motivation: these are characters of Gl_m. Recall this. Let V_λ be the irreducible representation of $\begin{cases} \mathfrak{gl}_m \\ GL_m \end{cases}$ with highest weight λ (it is f. dim). Let $N \geq m$. Then we have a repr of $\begin{cases} \mathfrak{gl}_N \\ GL_N \end{cases}$ with highest weight $\Lambda = (\lambda_1, \dots, \lambda_m, 0, \dots, 0)$

We have $y = \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_N \end{pmatrix} \in GL_N$

$$\chi_\Lambda(y) \overset{def}{=} tr_{V_\Lambda}(y)$$

<u>Ex.</u> $\lambda = (\lambda_1)$, $V_\lambda = S^{\lambda_1}\mathbb{C}^n$,

$$tr_{V_\lambda}(y) = \sum_{p_1 + \dots + p_N = \lambda_1} y_1^{p_1} \cdots y_N^{p_N} = h_{\lambda_1}(y).$$

<u>Thm.</u> $\chi_\Lambda(y) = S_\lambda(x)$, if $x_j = \frac{\sum_i y_i^j}{j}$

(we will not prove this theorem).

<u>Theorem.</u> $\sigma(v_{i_0} \wedge v_{i_1} \wedge \cdots) = S_\lambda(x) \in B^{(0)}$, where $\lambda = (i_0, i_1 + 1, i_2 + 2, \dots)$

<u>Pf.</u> It's easy to check that LHS and RHS have the same homogeneity degree. Let's denote

$$\sigma(v_{i_0} \wedge v_{i_1} \wedge \cdots) = P(x)$$

We want to show $P(x) = S_\lambda(x)$.

First we want to show that

$\forall y = (y_1, y_2, \dots)$

$$\langle \mathbb{1}, e^{y_1 a_1 + \dots + y_N a_N + \dots} P(x) \rangle =$$

$$= \langle \mathbb{1}, e^{y_1 a_1 + y_2 a_2 + \dots} S_\lambda(x) \rangle$$

This suffices since we can bring exponentials to the other side, getting $e^{\sum y_j a_{-j}}$, but the coefficients of this series are a basis of $\mathcal{B}^{(0)}$.

Now

$$\langle \mathbb{1}, e^{y_1 a_1 + y_2 a_2 + \dots} P(x) \rangle$$

$$= \langle \mathbb{1}, e^{y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + \dots} P(x) \rangle$$

$$= \langle \mathbb{1}, P(x+y) \rangle = P(y).$$

So, all we need to show is

$$\langle \mathbb{1}, e^{y_1 a_1 + \dots + y_N a_N \dots} P(x) \rangle = S_\lambda(y).$$

But this is

$$\langle \psi_0, e^{y_1 T + y_2 T^2 + \dots} (v_{i_0} \wedge v_{i_1} \wedge \dots) \rangle =$$

Now, $e^{y_1 T + y_2 T^2 + \cdots}$ is an element of GL_∞. (and clearly $GL(\infty)$ acts on $\mathcal{F}^{(0)}$).
So we need to compute the matrix of $e^{y_1 T + y_2 T^2 + \cdots}$

We have $e^{\sum y_j T^j} = \sum S_k(y) T^k$.

So the matrix corresponding to $e^{\sum y_j T^j}$ is

$$\begin{pmatrix} 1 & S_1(y) & S_2(y) & \cdots \\ & 1 & S_1(y) & \cdots \\ & & 1 & S_1(y) & \cdots \\ & & & 1 & S_1(y) \\ & & & & 1 & S_1(y) \\ & & & & & 1 \end{pmatrix}$$

Now we want to compute its matrix element in $\Lambda^{\frac{\infty}{2},0} V$.

In general, if $A \in GL(\infty)$, we have

$$\langle v_0 \wedge v_{-1} \wedge v_{-2} \ldots, A v_{i_0} \wedge v_{i_1} \wedge \ldots \rangle$$
$$= \det \left(A^{i_0, i_1, i_2, \ldots}_{0, -1, -2 \ldots} \right)$$

This is defined since the matrix above is of the form $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & /// \end{array} \right)$ (semiinfinite) finite

This is not exactly applicable in our case since $e^{\sum_i y_i T^i}$ lives in some completion of Gl_∞. But it still works since if $A = e^{y_1 T + y_2 T^2 + \cdots}$

then
$$A^{i_0\ i_1\ i_2\ \cdots}_{0,\,-1,\,-2} = \left(\begin{array}{c|c} \begin{matrix} 1 & & * \\ & 1 & \\ & & 1 \end{matrix} & * \\ \hline & * \end{array}\right)$$

so determinant is defined. And if we apply it, we get exactly the determinant defining the Schur polynomials. The theorem is proved.