Lectures 6-7

Representations of gl_∞.

Def. gl_∞ is the (Lie) algebra of all matrices $A = (a_{ij})$, $i, j \in \mathbb{Z}$, where almost all $a_{ij} = 0$.
(note that as an associative algebra gl_∞ does not have a unit).
Basis: E_{ij} — has 1 in ij-th position, 0 everywhere else.
This is an analog of gl_n.
Let us try to develop the representation theory of gl_∞ parallel to representation theory of gl_n

Vector representation: $V = \{v_j, j \in \mathbb{Z}\}$

$$v_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$$

Can define exterior powers $\Lambda^i V$, symmetric powers, and any Schur functors.

Also can define
<u>Highest weight representations</u>:

$$\mathfrak{gl}_\infty = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

upper triangular, diagonal, lower triangular

$\forall \lambda \in \mathfrak{h}^*$ can define Verma module M_λ, irreducible module $L_\lambda = M_\lambda / J_\lambda$,
J_λ, $J_\lambda = \mathrm{Ker}(\,,\,)$ - invariant form.
Also have a notion of unitarity:
$E_{ij}^\dagger = E_{ji}$, and V and Schur functors of V are unitary.

But the difference with finite dimensional case is that V and its Schur functors have no highest weight. So $\Lambda^i V$ and highest weight representations live in two different worlds. To marry these two worlds, we need to introduce <u>semiinf[illegible] wedge powers</u>. (important in QFT).

<u>Definition</u>. $\Lambda^{\infty/2} V$ is the free span of the vectors

$$v_{i_0} \wedge v_{i_1} \wedge \cdots$$

where $i_0, i_1, \ldots$ is a strictly decreasing sequence, such that $i_{k+1} = i_k - 1$ for large enough k (so it's a countably dimensional space).

We have $\Lambda^{\frac{\infty}{2}} V = \bigoplus_{m \in \mathbb{Z}} \Lambda^{\frac{\infty}{2}, m} V$, where $\Lambda^{\frac{\infty}{2}, m} V$ is the span of the wedges for which $i_k = -k + m$ for $k \gg 0$.

<u>Proposition</u>. The usual Leibniz rule defines a representation of $\mathfrak{gl}_\infty$ on $\Lambda^{\frac{\infty}{2}, m}$

$$E_{ij} v_{i_0} \wedge v_{i_1} \wedge \cdots = \begin{cases} v_{i_0} \wedge v_{i_1} \wedge \cdots \wedge \underset{\substack{\uparrow \\ k\text{-th if } i_k = j}}{v_i} \wedge \cdots \\ 0 \text{ if } j \neq i_k \ \forall k \end{cases}$$

reorder and put appropriate signs.

Example

$$E_{31}\, v_2 \wedge v_1 \wedge v_{-1} \wedge v_{-2} \cdots$$

$$= v_2 \wedge v_3 \wedge v_{-1} \wedge v_{-2} \cdots = -v_3 \wedge v_2 \wedge v_{-1} \wedge v_{-2} \cdots$$

Exercise. Check that it is a Lie algebra representation.

Proposition. $\Lambda^{\frac{\infty}{2}, m}$ is an irreducib. highest weight representation of gl_∞ with highest weight $\lambda = (\cdots 1 \cdots 1\ 0 \cdots 0 \cdots)_{-\infty}^{\infty} = \omega_m$, $m \in \mathbb{Z}$ and it is unitary (this is an analogue of $L_{\omega_m} = \Lambda^m V$ for gl_n).

Proof. we have vector

$w_m = v_m \wedge v_{m-1} \wedge \cdots$

Then $n_+ w_m = 0$, and $E_{ii} w_m = \lambda(E_{ii}) w_m$

Also w_m clearly generates the module.

Let us show that the module

is unitary (then it's automatically irreducible). To show this, it's enough to note that for the form in which the wedges are orthonormal, the operators E_{ij} satisfy $E_{ij} = E_{ji}^{\dagger}$. ∎

<u>Corollary</u>. Suppose $\lambda = (\lambda_i)$, $\lambda_i \in \mathbb{R}$, and $\lambda_i = \lambda_+$ for $i \gg 0$, $\lambda_i = \lambda_-$ for $i \ll 0$, and $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$. Then L_λ is unitary.

<u>Proof</u>. First suppose $\lambda_i = a$ $\forall i$. Then have 1-dim representation $X \to a \cdot \mathrm{Tr}\, X$, $a \in \mathbb{R}$ which is clearly unitary. Call this weight β_a.

We have: any λ as above has the form $\beta_a + \sum n_j \omega_j$, where $n_j \geq 0$ are integers. So L_λ is a summand in $L_{\beta_a} \otimes \bigotimes_i (L_{\omega_i})^{\otimes n_i}$, so is unitary.

Proposition. Any unitary representation L_μ of sl_2 has $\mu \in \mathbb{Z}_+$.

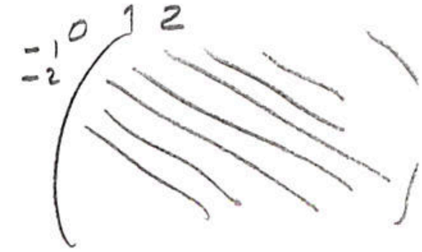
Proof. The form in degree n on L_μ is $n!(\mu-n+1)\cdots\mu$; if $\mu\in\mathbb{Z}_+$, this is ≥ 0, but if $\mu\notin\mathbb{Z}_+$, $L_\mu = M_\mu$ and this is sometimes negative.

Corollary. If an irreducible representation L_λ of gl_∞ is unitary then λ is as above. Suppose λ stabilizes at $+\infty$ and $-\infty$.

Proof. Consider the subrepresentation generated by v_λ over $sl_2 = \langle E_{i,i+1}, E_{i+1,i}\rangle$. Then the highest weight of this subrepresentation is $\mu = \lambda_i - \lambda_{i+1}$, so it must be in $\mathbb{Z}_+$.

Now we want to enlarge the algebra gl_∞.

Define $\overline{a}_\infty \supset gl_\infty$ to be the Lie algebra of all matrices with entries labeled by $\mathbb{Z}$ and finitely many nonzero diagonals.



For example, $\mathbb{1} \in \overline{\mathfrak{a}}_\infty$
$\notin \mathfrak{gl}_\infty$.

Grading: $\overline{\mathfrak{a}}_\infty = \bigoplus_{i\in\mathbb{Z}} \overline{\mathfrak{a}}_\infty^i$ ← nonzero entries only on the i-th digonal

One can think of $\overline{\mathfrak{a}}_\infty$ as an algebra of operators on sequences (column vectors). T – shift $(Tx)_n = x_{n+1}$. Elements of $\overline{\mathfrak{a}}_\infty$ are "difference operators

$$\sum_{k=p}^{a} a_n^{(k)} T^k$$

Let us now try to extend the representation ρ of $\mathfrak{gl}_\infty$ on $\Lambda^{\frac{\infty}{2}, m} V$ to $\overline{\mathfrak{a}}_\infty$. Try to extend "by continuity":

$$A \in \overline{\mathfrak{a}}_\infty^i \qquad A = \sum_{j\in\mathbb{Z}} a_j E_{j,j+i}$$

This is an infinite sum, but it becomes finite after acting on any semiinfinite wedge, if $i \neq 0$.
Indeed, if $i >> 0$, there won't be appropriate factors in the wedge,

while if $i \ll 0$, $E_{j,j+i}$ will produce a factor that is already present. The only problem arises if $j=0$. For instance $\rho(\mathbb{I})$ does not make sense. So we have to redefine ρ to extend it.

$$\hat{\rho}(E_{ij}) = \begin{cases} \rho(E_{ij}) & \text{unless } i=j,\ i\le 0 \\ \rho(E_{ij})-1, & i=j,\ i\le 0. \end{cases}$$

Then the map $\hat{\rho}$ can be extended to $\overline{a}_\infty$ by continuity. But it is not a homomorphism: in fact, $-\hat{\rho}([a,b]) + [\hat{\rho}(a), \hat{\rho}(b)]$ is a constant (so we get a representation of a central extension of $\overline{\mathfrak{a}}_\infty$).

Let us describe this constant more explicitly.

Namely, consider $A, B \in \overline{\mathfrak{a}}_\infty$, and write them as block 2 by 2 matrices,

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \quad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right),$$ where

the division corresponds to $i\le 0$ and $i>0$.

<u>Proposition</u>. The formula

$$\alpha(A,B) = tr(-B_{12}A_{21} + A_{12}B_{21}) = tr(-A_{21}B_{12} + B_{21}A_{12})$$

defines a 2-cocycle on $\overline{\mathfrak{gl}}_\infty$, which is nontrivial.

Note that this is well defined since $A_{21}, A_{12}, B_{21}, B_{12}$ have only finitely many nonzero entries

<u>Remark</u>

For A, B in $\mathfrak{gl}_\infty$, $\alpha(A,B) = Tr(J[A,B])$, where $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, since

$$[A,B] = \begin{pmatrix} [A_{11}, B_{11}] + A_{12}B_{21} - B_{12}A_{21} & * \\ * & * \end{pmatrix}$$

This implies that $\alpha|_{\mathfrak{gl}_\infty}$ is a trivial cocycle.

<u>Pf</u>. The fact that α is a cocycle will follow from the following proposition

Nontriviality: restrict to

$\mathcal{A} = \langle T^i \rangle$, $T = \begin{pmatrix} \ddots & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\quad T^i = a_i$

Then $\alpha(T^i, T^j) = i\delta_{i,-j}$, so get the Heisenberg extension which is nontrivial

Theorem.

$$-\hat{\rho}([A,B]) + [\hat{\rho}(A), \hat{\rho}(B)] = \alpha(A,B).$$

Pf Homework problem

So we get a central extension

$\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}K$,

$[(A,\alpha),(B,\beta)] = ([A,B], \alpha(A,B))$, and

$\hat{\rho}$ extends to a representation of this central extension, with $\hat{\rho}(K) = 1$.

Now all the theory can be extended

In particular, we have representations $L_{\sum n_i \omega_i}$ which are unitary representations of level (i.e K-eigenvalue) $\sum_i n_i$.

We have seen that $\mathcal{A} \hookrightarrow \mathfrak{a}_\infty$.

In fact, we also have $Vir \hookrightarrow \mathfrak{a}_\infty$.

Namely, we can assume that $V = V_{\alpha,\beta}$.

Recall that $L_m v_k = (k - \alpha - \beta(m+1)) v_{k-m}$

(we change numbering)

$k \to -k$

Then

$$[L_n, L_m] = (n-m) L_{n+m} + \alpha(L_n, L_m)$$

$$\alpha(L_n, L_m) = \delta_{n,-m}\left(\frac{n^3-n}{12} c_\beta + 2n h_{\alpha,\beta}\right)$$

$$c_\beta = -12\beta^2 + 12\beta - 2$$

$$h_{\alpha,\beta} = \frac{1}{2}\alpha(\alpha + 2\beta - 1).$$

This is almost Vir, except for $h_{\alpha,\beta}$. $L_0 \to L_0 + h_{\alpha,\beta}$, get a Vir representation with $c_\beta = -12\beta^2 + 12\beta - 2$ (not always unitary)

In fact, if $\psi_m = v_m \wedge v_{m-1} \wedge \cdots$

then $L_0 \psi_m = h_m \psi_m$,

$$h_m = \frac{1}{2}(\alpha - m)(\alpha + 2\beta - 1 - m)$$

(exercise).