Lecture 5.

Also, for any $\lambda \in \mathbb{C}$, we have a deformed action of Vir, depending on λ, and giving the above action for $\lambda = 0$.

Theorem. The formulas

$$\tilde{L}_k = \frac{1}{2} \sum_{j \in \mathbb{Z}} :a_j a_{k-j}: + i\lambda k a_k, \qquad \tilde{L}_0 = \frac{\lambda^2 + a_0^2}{2} + \sum_{j>0} a_{-j} a_j$$

define an action of Vir on F_μ which has $c = 1 + 12\lambda^2$.

Note that $\left[\tilde{L}_m, a_n\right] = -n a_{m+n} + i\lambda n^2 \delta_{m,-n}$

Proof: Homework

Quantum fields.

Physicists write the formulas like the ones above in terms of quantum fields. In mathematics, this fits into the formalism of vertex operator algebras.

Namely, for the oscillator algebra $\mathcal{A}$, define the quantum field

$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$. This is just a formal series which is a generating function for the generators. It's a series infinite in both directions, but $\forall v \in F_\mu$, $a(z)v$ is infinite only in the positive direction.

Consider the series $\sum_{n\in\mathbb{Z}} z^{-n-1} w^{n}$ and denote it $\delta(w-z)$.

Motivation: if f is a Laurent polynomial, then the formal integral $\frac{1}{2\pi i}\int_{|z|=1} \delta(w-z) f(z)\,dz = f(w)$.

So $$[a(z), a(w)] = \sum_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1} = \delta'(w-z).$$ in F_μ.

Also $$a(z)a(w) - :a(z)a(w): = \sum_{n\geq 0} [a_n, a_{-n}] z^{-n-1} w^{n}$$
$$= \sum_{n>0} n z^{n-1} w^{-n-1} = \frac{1}{z^2} \cdot \frac{1}{\left(1-\frac{w}{z}\right)^2} = \frac{1}{(z-w)^2}$$

So $$a(z)a(w) = :a(z)a(w): + \frac{1}{(z-w)^2}.$$

We see that $:a(z)a(w):$ represents "the regular part" and $\frac{1}{(z-w)^2}$ "the singular part"

We can do a similar thing with the Virasoro algebra. We define

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

In the Witt algebra:

$$[T(z), T(w)] = \sum (n-m) L_{n+m} z^{-n-2} w^{-m-2}$$

$$= \sum_k L_k \sum_n z^{-k+m-2} w^{-m-2} (k-2m)$$

$n=k-m$

$$= -\left(\sum L_k z^{-k-2}\right) \cdot \sum_m (-2m-2) z^{m} w^{-m-2}$$

$$+ \sum L_k (-k+2) \cdot z^{-k-3} \cdot \sum_n z^{m+1} w^{-m-2}$$

$$= 2T(z)\,\delta'(w-z) - T'(z)\,\delta(w-z).$$

The central extension term gives

$$\sum_n \frac{n^3-n}{12} C \cdot z^{-n-2} w^{n-2} =$$

$$= \frac{C}{12}\,\delta'''(w-z).$$ So for Vir, we have

$$[T(z),T(w)] = 2T(z)\delta'(w-z) - T'(z)\delta(w-z) + \frac{c}{12}\delta'''(w-z)$$

also

$$[T(z), a(w)] = \sum(-m a_{n+m}) z^{-n-2} w^{-m-1}$$

$$= \sum_{\substack{k \\ \| \\ m+n}} a_k \sum_m (-m) z^{m-k-2} w^{-m-1}$$

$$= \Big(\sum_k a_k z^{-k-1}\Big)\Big(\sum_m (-m) z^{m-1} w^{-m-1}\Big)$$

$$= a(z)\delta'(w-z).$$

Finally, the representation of Vir on F_μ (for $\lambda = 0$) is given by the formula

$$T(z) = \frac{1}{2} :a(z)^2:$$

Exercise 1 Show that the formula

$$(*)\ T(z) = \frac{1}{2} :a(z)^2: + \beta\, \partial a(z), \quad \beta \in \mathbb{C},$$

defines a representation of Vir on F_μ with $C = 1 - 12\beta^2$.

<u>Exercise 2</u>. Show that $\forall \beta \in \mathbb{C}$ the formulas

$$L_n \mapsto L_n + \beta a_n, \; n \neq 0$$
$$L_0 \mapsto L_0 + \beta a_0 + \frac{\beta^2}{2} K$$
$$C \to C$$

defines a splitting $\varphi_\beta: Vir \longrightarrow Vir \ltimes \mathcal{A}$.
Show that for $\beta = i\lambda$ φ_β transforms the repr. $(*)$ into the repr from homework 2 (which is unitary for $\lambda \in \mathbb{R}$).

see below
Now let us go back to repr theory.

<u>Proposition</u>. If $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, the operators $\tilde{L}_n$ define a unitary representation of Vir, with respect to the usual Hermitian structure on F_μ.

Before we prove this, we need

<u>Prop.</u> If $\mu \in \mathbb{R}$, F_μ is a unitary representation of $\mathcal{A}$, with the usual Hermitian form.

<u>Proof.</u> The Hermitian form has the form

$$\langle x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}, x_1^{m_1} \cdots x_k^{m_k} \rangle = \prod \delta_{n_i m_i} \cdot$$

$\cdot\, n_1! \cdots n_k!\, 1^{n_1} \cdots k^{n_k}$, so the form is <u>positive definite</u>.

Now we can prove the above proposition. The proof is easy, since we see from the formulas that

$L_k^+ = L_{-k}$ (using that $i^+ = -i$).
$i = \sqrt{-1}$

<u>Prop.</u> Let V be a unitary representation in category $\mathcal{O}^+$ over a graded Lie algebra $\mathfrak{g}$. Then V is a direct sum of irreducible representations

Proof. Lemma. Let V be be a highest weight unitary representation. Then V is irreducible.

Pf. Suppose V has highest weight λ. Then we have a projection $V \twoheadrightarrow L_\lambda$. (I will write L_λ instead of L_λ^+ when no confusion is possible). Let K be the kernel of this projection, and $K^\perp$ be the orthogonal complement. Then the projection $K^\perp \to L_\lambda$ is an isomorphism, so $V \cong L_\lambda \oplus K$. Thus, $K = 0$ (as V is generated by it highest weight vector). Lemma is proved

Now we can prove the proposition. Let $v \in V$ be a highest degree vector, and $\langle v \rangle \subset V$ be the representation generated by V. Then $\langle v \rangle \cong L_{\lambda_1}$, and $V \cong L_{\lambda_1} \oplus V_1$. Then apply the same argument to V_1', etc. Since degrees in V lie infinitely many arithmetic progressions, this process is exhaustive, and $V \cong L_{\lambda_1} \oplus L_{\lambda_2} \oplus \cdots$

Proposition is proved.

<u>Cor</u> F_μ is a completely reducible representation of Vir. for $\lambda, \mu \in \mathbb{R}$.

<u>Remark</u>. It is easy to see that actually for generic μ, F_μ is irreducible, but even if it's reducible, it is completely reducible, i.e. is not a highest weight module

$$\boxed{F_\mu \cong M_{\frac{\mu^2+\lambda^2}{2},\, 1+12\lambda^2}}$$

So we see that

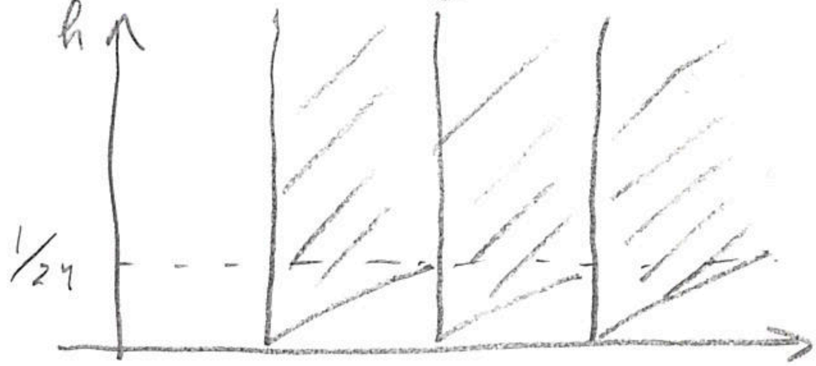
$L_{h,c}$ is unitary if $c \geq 1$ and $h \geq \frac{c-1}{24}$.

Also $L_{0,1}$ is unitary, so by using tensor product, we see that $L_{h,c}$ is unitary if

$c \geq m$ and $h \geq \frac{c-m}{24}$ $\forall m \geq 1$.

($L_{h,c}$ is a summand in $L_{0,1}^{\otimes m} \otimes L_{h,c-m}$).

So we see that representations in this region are unitary:



What happens for $c<1$?

Introduce "Free fermions" algebra (Clifford algebra) C_δ, $\delta \in \{0, \frac{1}{2}\}$, generated by ψ_m, $m \in \delta + \mathbb{Z}$, with

$$\psi_m \psi_n + \psi_n \psi_m = \delta_{m,-n}.$$

C_0 is called the Ramond sector and $C_{\frac{1}{2}}$ the Neveu-Schwarz sector in physics.

We have a representation of C_δ on polynomials in anticommuting variables: ξ_i, $V_\delta = \Lambda(\xi_n, n \geq 0)$, $n \in \delta + \mathbb{Z}$

via $\psi_{-n} \mapsto \xi_n$, $n > 0$

$\psi_n \mapsto \frac{\partial}{\partial \xi_n}$, $\quad \psi_0 \to \frac{1}{\sqrt{2}} \left(\xi_0 + \frac{\partial}{\partial \xi_0}\right)$.

Proposition.

Let $$L_k = \delta_{k,0}\frac{1-2\delta}{16} + \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j:\psi_{-j}\psi_{j+k}:,$$

Where $$:\psi_n\psi_m: = \begin{cases} \psi_n\psi_m, & m\geq n \\ -\psi_m\psi_n, & m\leq n \end{cases}$$

Then

(1) $[\psi_m, L_k] = (m+\frac{k}{2})\psi_{m+k}$

(2) $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{m^3-m}{24}$

So $$L_0 = \frac{1-2\delta}{16} + \sum_{j>0} j\psi_{-j}\psi_j, \quad c=\frac{1}{2}.$$

So we can consider the correspon-ding representation V_δ of Vir

It's not irreducible:

$V_\delta = V_\delta^+ \oplus V_\delta^-$, where V_δ^+ is the even part and V_δ^- is the odd part and they are irreducible (this is nontrivial)

If $\delta = 0$, $V_0^+ = L_{\frac{1}{16},\frac{1}{2}}$, $V_0^- = L_{\frac{1}{16},\frac{1}{2}}$

~~whate~~ so

$$ch\, L_{\frac{1}{16},\frac{1}{2}} = q^{\frac{1}{16}} \prod_{n \geq 1} (1+q^{n+1}).$$

If $\delta = 1$, $V_{1/2}^+ = L_{0,\frac{1}{2}}$, $V_{1/2}^- = L_{\frac{1}{2},\frac{1}{2}}$

so

$$ch\, L_{0,\frac{1}{2}} = \underset{\text{part}}{\text{Integer}}\left(\prod (1+q^{n+\frac{1}{2}})\right)$$

$$ch\, L_{\frac{1}{2},\frac{1}{2}} = \underset{\text{part}}{\overset{\text{Half}}{\text{integer}}}\left(\prod (1+q^{n+\frac{1}{2}})\right)$$

(where (half) integer part is the sum of all (half) integer powers of q.