lecture 5.
Also for any \( \lambda \in \mathbb{C} \), we have a deformed action of \( \text{Vir} \), depending on \( \lambda \), and giving the above action for \( \lambda = 0 \).

\[ L_{\lambda} \]

**Theorem.** The formulas

\[ L_{\lambda} = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_j a_{\lambda-j} + i \lambda k a_0 + \sum_{j \geq 0} a_j a_{\lambda-j} \]

define an action of \( \text{Vir} \) on \( F \), which has \( c = 1 + 2\lambda^2 \).

Note that \( [L_m, a_n] = -m a_{m+n} + i \lambda n^2 \delta_{m-n} \).

**Proof:** Homework

**Quantum fields.**

Physicists write the formulas like the ones above in terms of quantum fields. In mathematics, this fits into the formalism of vertex operator algebras.

Namely, for the oscillator algebra \( \mathcal{A} \), define the quantum field

\[ a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \]. This is just a formal
series which is a generating function for the generators. It's a series infinite in both directions, but $\forall v \in \mathbb{F}_n$, $a(z)v$ is infinite only in the positive direction.

Consider the series $\sum_{n \in \mathbb{Z}} z^{-n}w^n$ and denote it $\delta{(w - z)}$.

Motivation: if $f$ is a Laurent polynomial, then the formal integral
\[
\frac{1}{2\pi i} \int_{|z|=1} \delta{(w-z)} f(z) \, dz = f(w).
\]

So
\[
[a(z), a(w)] = \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1} = \delta'(w-z).
\]

Also
\[
a(z)a(w) - [a(z), a(w)] = \sum_{n \geq 0} [a_n, a_{q-n}] z^{-n} w^n
\]
\[
= \sum_{n \geq 0} n z^{-n-1} w^{-n-1} = \frac{1}{z^2} \frac{1}{\left(1 - \frac{w}{z}\right)^2} = \frac{1}{(z-w)^2}
\]

So
\[
a(z)a(w) = [a(z), a(w)] + \frac{1}{(z-w)^2}.
\]

We see that $[a(z), a(w)]$ represents "the regular part" and $\frac{1}{(z-w)^2}$ "the singular part".
We can do a similar thing with the Virasoro algebra. We define

\[ T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \]

In the Witt algebra,

\[ [T(z), T(w)] = \sum_{n=-m}^{n=m} L_{n+m} z^{-n-1} w^{-m-2} \]

\[ = \sum_{k=1}^{k} \sum_{n} \frac{z^{k+m-2}}{n} w^{-m-2} \]

\[ = \left( \sum_{m} L_{m} z^{-k-2} \right) \sum_{m} (z^{-2m-2}) \sum_{n} z^{-m+1} w^{-m-2} \]

\[ + \sum_{k=1}^{k} \sum_{n} L_{k} (-k+2) \frac{z^{-k-3}}{n} \sum_{n} \frac{z^{-m+1}}{n} w^{-m-2} \]

\[ = 2T(z) \delta'(w-z) - T'(z) \delta(w-z). \]

The central extension term gives

\[ \sum_{n} \frac{n^3-n}{12} C \cdot z^{-n-2} w^{-n-2} = \]

\[ = \frac{C}{12} \delta'''(w-z). \] So for \( \text{Vir} \), we have...
\[ \begin{align*}
[T(z), T(w)] &= 2 T(z) \delta'(w-z) - T'(z) \delta(w-z) \\
&+ \frac{c}{12} \delta'''(w-z)
\end{align*} \]

Also
\[ \begin{align*}
[T(z), a(w)] &= \sum (-m a_{m+n}) z^{-n-2} w^{-m-1} \\
&= \sum_{k} \sum_{m} (-m) z^{m-k-2} w^{-m-1} \\
&= \left( \sum_{k} a_{k} z^{-k-1} \right) \left( \sum_{m} (-m) z^{m-1} w^{-m-1} \right) \\
&= a(z) \delta'(w-z).
\end{align*} \]

Finally, the representation of Vir on \( F_{\lambda} \) (for \( \lambda = 0 \)) is given by the formula
\[ T'(z) = \frac{1}{2} : a(z)^2 : \]

exercise 1 show that the formula
\[ (*) \ T(z) = \frac{1}{2} : a(z)^2 : + \beta \mathfrak{d} a(z) , \beta \in \mathbb{C}, \]
defines a representation of Vir on \( F_{\lambda} \) with \( C = 1 - 12 \beta^2 \).
Exercise 2. Show that the formulas
\[ L_n \mapsto L_n + \beta a_n, \quad n \neq 0 \]
\[ L_0 \mapsto L_0 + \beta a_0 + \frac{\beta^2}{2} K \]
\[ C \mapsto C \]
defines a splitting \( \Psi_\beta : \text{Vir} \rightarrow \text{Vir} \times \mathbb{A} \).
Show that for \( \beta = i \lambda \) \( \Psi_\beta \) transforms the repr. (\( \lambda_\lambda \)) into the repr. from Homework 2 (which is unitary for \( \lambda \in \mathbb{R} \)). See below.

Now let us go back to rep. theory.

**Proposition.** If \( \lambda \in \mathbb{R} \), the operators \( L_n \) define a unitary representation of \( \text{Vir} \), with respect to the usual Hermitian structure on \( \mathbb{F}_0 \).
Before we prove this, we need Prop. If \( \mu \in \mathbb{R} \), \( E_n \) is a unitary representation of \( A \), with the usual Hermitian form.

Proof. The Hermitian form has the form
\[
\langle x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}, x_1^{m_1} \cdots x_k^{m_k} \rangle = T \delta_{n,m}
\]
so the form is positive definite.

Now we can prove the above proposition. The proof is easy, since we see from the formulas that
\[
L_k^+ = L_{-k} \quad \text{(using that } i^+ = -i, \quad i = \sqrt{-1})
\]

Prop. Let \( V \) be a unitary representation in category \( \mathcal{O} \) \( \mathcal{O}^+ \) over a graded Lie algebra \( g \). Then \( V \) is a direct sum of irreducible representations.
Lemma. Let \( V \) be a highest weight unitary representation. Then \( V \) is irreducible.

**Proof.** Suppose \( V \) has highest weight \( \lambda \). Then we have a projection \( V \to L_\lambda \).

(I will write \( L_\lambda \) instead of \( L_\lambda^+ \) when no confusion is possible.) Let \( K \) be the kernel of this projection, and \( K^\perp \) be the orthogonal complement. Then the projection \( K^\perp \to L_\lambda \) is an isomorphism, so \( V \cong L_\lambda \oplus K \).

Thus, \( K = 0 \) (as \( V \) is generated by its highest weight vector). Lemma is proved.

Now we can prove the proposition. Let \( v \in V \) be a highest weight vector, and \( \langle v \rangle \subseteq V \) be the representation generated by \( v \).

Then \( \langle v \rangle \cong L_{\lambda_1} \), and \( V \cong L_{\lambda_1} \oplus V_1 \).

Then apply the same argument to \( V_1 \), etc. Since degrees in \( V \) lie
In infinitely many arithmetic progressions, this process is exhaustive, and \( V \cong L_{x_1} \oplus L_{x_2} \oplus \ldots \notag \).

Proposition is proved.

\( \text{Cor. } F_{\mu} \) is a completely reducible representation of \( \text{Vir}_1 \) for \( \mu \in \mathbb{R} \).

Remark. It is easy to see that actually, for generic \( \mu \), \( F_{\mu} \) is irreducible, but even if it's reducible, it is completely reducible, i.e., is not a highest weight module.

\[
F_{\mu} \cong M_{\frac{\mu^2 + \lambda^2}{2}, \frac{\lambda^2}{2}}
\]

So we see that \( L_{h, c} \) is unitary if \( c \geq 1 \) and \( h \geq \frac{c-1}{2c} \).

Also \( L_{0, 1} \) is unitary, so by using tensor product, we see that \( L_{h, c} \) is unitary if \( c \geq m \) and \( h \geq \frac{c-m}{2c} \) \( \forall m \geq 1 \).

\( L_{h, c} \) is a summand of \( L_{0, 1} \otimes L_{h, c} \) \( \leq m \).
So we see that representations in this region are unitary.

What happens for \( c < 1 \)? Introduce "Free fermions" algebra \((\mathcal{F}_c, \delta = \{0, \frac{1}{2}, \frac{3}{2}\})\), generated by \( \psi_m, m \in \delta + \mathbb{Z} \), with

\[
\psi_m \psi_n + \psi_n \psi_m = \delta_{m,-n}.
\]

\( C_0 \) is called the Ramond sector and \( C_{\frac{1}{2}} \) the Neveu-Schwarz sector in physics.

We have a representation of \( \mathcal{F}_c \) on polynomials in anticommuting variables: \( \frac{3}{2} \), \( \psi_c = \Lambda (\frac{3}{2} n, n \geq 0) \), \( n \in \delta + \mathbb{Z} \)

via \( \psi_n \mapsto \frac{3}{2} n, \quad n > 0 \)

\( \psi_n \mapsto \frac{\partial}{\partial \bar{z}_n} \), \( \quad \psi_0 \mapsto \frac{1}{\sqrt{2}} (\bar{z}_0 + \frac{3}{2} \bar{z}_0) \).
Proposition

Let \( L_k \) be defined as
\[
L_k = \delta_k, 0 \left( 1 - 2\delta \right) + \frac{1}{2} \sum_{j \in \delta + 2Z} \delta^j : \psi_j - \psi_{j+k},
\]

where \( \psi_n : \psi_m = \sum_{j=0}^{\min(n, m)} L - \psi_{j-n} \psi_j, m \geq n \)

Then

1. \( [\psi_m, L_k] = (m + \frac{k}{2}) \psi_{m+k} \)
2. \( [L_n, L_m] = (n-m) L_{m+n} + \delta_{n-m} \frac{m^3 - m}{2} \)

So
\[
L_0 = \frac{1 - 2\delta}{16} + \sum_{j > 0} j \psi_j \psi_j, \quad C = \frac{1}{2}.
\]
So we can consider the corresponding representation $V_\delta$ of $Vir$.

It's not irreducible:

$V_\delta = V_\delta^+ \oplus V_\delta^-$, where $V_\delta^+$ is the even part and $V_\delta^-$ is the odd part, and they are $V_\delta$ irreducible (this is nontrivial).

If $\delta = 0$, $V_0^+ = \mathbb{I} \frac{1}{16}, 0, \frac{1}{2}$, $V_0^- = \mathbb{I} \frac{1}{16}, 0, \frac{1}{2}$.

Where $\delta = 0$,

$$
\text{ch} \ L_0 = \begin{cases} 
\frac{1}{16, \frac{1}{2}} & n \geq 1 \\
\prod (1 + q^{n+1}) & n \geq 1 
\end{cases}
$$

If $\delta = 1$, $V_{\frac{1}{2}}^+ = \mathbb{I} 0, \frac{1}{2}$, $V_{-\frac{1}{2}} = \mathbb{I} \frac{1}{2}, 0, \frac{1}{2}$.

So,

$$
\text{ch} L_{0, \frac{1}{2}} = \text{integer part} \left( \prod \left( 1 + q^{n+\frac{1}{2}} \right) \right)
$$

$$
\text{ch} L_{\frac{1}{2}, \frac{1}{2}} = \text{half integer part} \left( \prod \left( 1 + q^{n+\frac{1}{2}} \right) \right)
$$

(where integer part is the sum of all (half) integer powers of $q$.)