Lecture 4

Examples. 1. Let $\mathfrak{g} = sl(2) = \langle e, f, h \rangle$,
$[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$,
$\deg(h) = 0$, $\deg(e) = 1$, $\deg(f) = -1$, so $\mathfrak{h} = \langle h \rangle$
Then $M_\lambda^+ = \langle f^n v_\lambda^+ \rangle$, $M_\lambda^- = \langle e^n v_\lambda^- \rangle$,
where $\lambda \in \mathfrak{h}^*$. We can identify $\mathfrak{h}^*$ with $\mathbb{C}$
by $\lambda \mapsto \lambda(h)$; so we'll regard λ as a number.
We have

$$(f^n v_\lambda^+, e^n v_{-\lambda}^-) = (f^n, e^n)_{\lambda, n} =$$

$$= ((-1)^n e^n f^n v_\lambda^+, v_{-\lambda}^-) = (-1)^n n! \lambda(\lambda-1)\cdots(\lambda-n+1).$$

So if $\lambda \notin \mathbb{Z}_+$, M_λ^+ is irreducible;
if $\lambda \in \mathbb{Z}_+$, then $\operatorname{Ker}(\,,\,) = \langle f^n v_\lambda^+ \rangle_{n \geq \lambda+1} = J_\lambda^+$

so $M_\lambda^+ / J_\lambda^+ = L_\lambda^+ = \langle v_\lambda^+, f v_\lambda^+, \ldots, f^\lambda v_\lambda^+ \rangle$
is the $\lambda+1$-dimensional irreducible $sl(2)$-module.

2. Consider the Virasoro algebra $\mathfrak{g} = Vir$.
We have $\mathfrak{g}_0 = \langle L_0, C \rangle$, so $\mathfrak{g}_0^* = \{(h,c)\}$ $= \lambda$
$h = \lambda(L_0)$, $c = \lambda(C)$ (c is called central charge).

$v = v_{h,c}$

In degree 1:

$$(L_{-1}, L_{-1})_{h,c,1} = \,?$$

We need to compute $-L_1 L_{-1} v_{h,c}$, $v = v_{h,c}$.

It's easy to see that this is

$-2L_0 v_{h,c} = -2hv$. So $\det_1 = -2h$, vanishes when $h=0$.

Now consider degree 2.

We have $L_0 L_{-1} v = (h+1) L_{-1} v$

$$L_1^2 L_{-1}^2 v = L_1 \cdot 2L_0 \cdot L_{-1} v + L_1 L_{-1} L_1 L_{-1} v$$

$$= 2(h+1) L_1 L_{-1} v + (L_1 L_{-1})^2 v = \left(\begin{matrix}4h^2+4h^2 \\ +4h\end{matrix}\right) v = (8h^2+4h)v$$

$$L_2 L_{-1}^2 v = 3 L_1 L_{-1} v = 6hv$$

$$L_1^2 L_{-2} v = 3 L_1 L_{-1} v = 6hv$$

$$L_2 L_{-2} v = \left(4h + \tfrac{1}{2}c\right) v,$$

So we get matrix of $(\,,\,)_{h,c,2}$ in basis $L_{-1}^2 v_+$ and $L_{-2} v_+$; $L_1^2 v_-$ and $L_2 v_-$:

$$\begin{pmatrix} 8h^2+4h & 6h \\ -6h & -4h-\frac{1}{2}c \end{pmatrix}$$

The determinant of this matrix is

$$\det_2 = -\left(4h(2h+1)(4h+\tfrac{1}{2}c) - 36h^2\right)$$

$$= -4h\left((2h+1)(4h+\tfrac{1}{2}c) - 9h\right)$$

So this vanishes on a union of a line and a hyperbola.

The ~~dual~~ restricted ~~dual~~ module.

If M is a graded vector space, $M = \oplus M[i]$, then the restricted dual $M^\vee \subset M^*$ is $\oplus M[i]^*$. We have $M^{\vee\vee} = M$ if M has f.d. homogeneous subspaces.

Prop. We have inverse antiequivalences $\mathcal{O}^+ \xrightarrow{\vee} \mathcal{O}^-$, $\mathcal{O}^- \xrightarrow{\vee} \mathcal{O}^+$ given by taking restricted duals.

Pf. Clear

Prop. $\forall \lambda$, we have a natural homomorphism $M_\lambda^+ \to (M_{-\lambda}^-)^\vee$, given by the invariant form. Its kernel is J_λ^+, so it is an isomorphism iff $\mathrm{Ker}(\,,\,) = 0$, i.e. iff M_λ^+ is irreducible.

<u>Singular vectors</u>. It follows from the above that if M_λ^+ is not irreducible then for some $d>0$ it has a vector $w \neq 0$, $w \in M_\lambda^+$ of degree $-d$ which is killed by $\mathfrak{n}_+$, and is an eigenvector of $\mathfrak{h}$ with some eigenvalue $\mu \in \mathfrak{h}^*$. Such a vector is called a singular vector. The smallest d for which there is such a vector is the smallest d for which $\det(\,,\,)_{\lambda,d} = 0$. Note that for any $\mathfrak{g}$-module Y, $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, Y)$ $= Y_\lambda^{\mathfrak{n}_+}$, the $\mathfrak{h}$-eigenspace with eigenvalue λ in $Y^{\mathfrak{n}_+}$. So every singular vector w defines a graded homomorphism $\varphi : M_\mu^+ \to M_\lambda^+$, and vice versa.

More generally, we can define a notion of a singular vector in any graded module from $\mathcal{O}^+$.

Involutions. In many cases, $\mathfrak{g}$ has an involutive automorphism $\omega: \mathfrak{g} \to \mathfrak{g}$ $\omega^2 = 1$, $\omega|_{\mathfrak{g}_0} = 0_{-1}$ such that $\omega(\mathfrak{g}_i) = \mathfrak{g}_{-i}$, In this case, we have an equivalence $\mathcal{O}^+ \to \mathcal{O}^-$ given by $M \to M^\omega$, so composition with v gives an anti autoequivalence $\mathcal{O}^+ \to \mathcal{O}^+$ denoted by c (contragredient module, $M \mapsto M^c$). Also, in this case we can consider $(,)$ as a bilinear form on M_λ^+, as $(M_{-\lambda}^-)^\omega = M_\lambda^+$, but the form is contravariant: $(av, w) = -(v, \omega(a)w)$. Such a form is called a Shapovalov form, which is symmetric, and its properties are similar to the properties considered before. Any highest weight module carries a Shapovalov form.

Examples: $\omega: \mathcal{A} \to \mathcal{A}$, $\omega(a_i) = -a_{-i}$, $\omega(K) = -K$

$\omega: sl_2 \to sl_2$, $\omega(e) = f$, $\omega(f) = e$, $\omega(h) = -h$.

$\omega: Vir \to Vir$, $\omega(L_i) = -L_{-i}$, $\omega(C) = -C$

$\omega : \overset{\swarrow \text{simple}}{\mathfrak{g}} \to \mathfrak{g}$ $\quad \omega(e_i) = f_i, \ \omega(f_i) = e_i$
$\omega(h_i) = -h_i$

$\omega : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}, \quad \omega(at) = \omega(a)t^{-1}$
$\omega(K) = -K.$

<u>Unitary structures</u>. Let $\mathfrak{g}$ be a complex Lie algebra, and $\dagger : \mathfrak{g} \to \mathfrak{g}$ an antilinear antiinvolution.

<u>Exercise</u>. Let $\mathfrak{g}_{\mathbb{R}} = \{x \in \mathfrak{g} \mid x^\dagger = -x\}$. Then $\mathfrak{g}_{\mathbb{R}}$ is an $\mathbb{R}$-Lie algebra, and $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathfrak{g}$.

<u>Def</u>. A $\mathfrak{g}$-module V is called Hermitian if it is equipped with a nondegenerate Hermitian form $(\ ,\ )$ such that $(av, w) = (v, a^\dagger w)$. It is unitary if this form is positive definite.

If $\mathfrak{g}$ is a nondegenerate Lie algebra, we will consider unitary structures $\dagger : \mathfrak{g} \to \mathfrak{g}$ which map $\mathfrak{g}_i$ to $\mathfrak{g}_{-i}$, and if $\lambda \in \mathfrak{g}_{0,\mathbb{R}}^*$ then M_λ^+ has a Hermitian form since $(M^-_{-\lambda})^\dagger = M^+_\lambda$. Moreover, any highest weight module with $\lambda \in \mathfrak{g}^*_{0\mathbb{R}}$ carries a Hermitian form, and L^+_λ carries a nondegenerate one.

<u>Ex.</u> $\dagger: \mathcal{A} \to \mathcal{A}$, $a_i^\dagger = a_{-i}$, $K^\dagger = K$

$\dagger: sl_2 \to sl_2$, $e_i^\dagger = f_i$, $f_i^\dagger = e_i$, $h_i^\dagger = h_i$
or $\mathfrak{g} \to \mathfrak{g}$
(for $\mathfrak{g}$ simple)

$\dagger: \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ $(at)^\dagger = a^\dagger t^{-1}$

$\dagger: Vir \to Vir$, $L_i^\dagger = L_{-i}$, $C^\dagger = C$

So e.g. $L_{h,c}$ for real h, c carries a Hermitian form, and one may ask when it is positive. This is a nontrivial and interesting question for which a complete answer is known, and we'll discuss it.

Representations of Vir on F_μ

Recall that we have a semidirect product $W \ltimes \mathcal{A}$, and an action of $\mathcal{A}$ on F_μ.

Question: Can we extend it to an action of W? I.e.

1) Can we find operators $L_n : F_\mu \circlearrowleft$ such that $[L_n, a_m] = -m a_{n+m}$?

2) Do they satisfy the W-relations.

Aswers! 1) yes, and uniquely up to adding a constant

2) No, but almost yes.

Pf of uniqueness: $[L_n' - L_n'', a_m] = 0$, so by Dixmier's lemma, $L_n' - L_n''$ is a constant.

Pf of existence: Let $:a_i a_j: = \begin{cases} a_i a_j, & j \geq i \\ a_j a_i, & j \leq i \end{cases}$

Set $L_i = \frac{1}{2} \sum_{j \in \mathbb{Z}} :a_j a_{i-j}:$

It is easy to see that these are well defined on F_μ, and

$[L_n, a_m] = -m a_{n+m}$.

Now $[L_i, L_j] - (i-j) L_{i+j}$ commutes with a_i, so by Schur - Dixmier it is a constant. But if $i+j \neq 0$, this has nonzero degree, so it is 0.

On the other hand

$[L_i, L_{-i}] - 2iL_0 = K_i$ is a constant. We compute: $L_0 \mathbb{1} = (\frac{\mu^2}{2}+E)\mathbb{1} = \frac{\mu^2}{2}\mathbb{1}$

$$[L_1, L_{-1}] \cdot \mathbb{1} = L_1 L_{-1} \mathbb{1} = \mu L_1 a_{-1} \mathbb{1}$$

$$\mu^2 a_1 a_{-1} \cdot \mathbb{1} = \mu^2 \cdot \mathbb{1} = 2L_0 \mathbb{1}.$$

$$[L_2, L_{-2}] \mathbb{1} = L_2 L_{-2} \mathbb{1} = \mu^2 a_2 a_{-2} \mathbb{1}$$

$$+\frac{1}{4} a_1^2 a_{-1}^2 \mathbb{1} = 2\mu^2 \mathbb{1} + \frac{1}{4}\left(\frac{\partial}{\partial x}\right)^2 x^2 \mathbb{1} = \left(2\mu^2 + \frac{1}{2}\right)\mathbb{1}$$

$$= \left(4L_0 + \frac{1}{2}\right)\mathbb{1}.$$ This shows that

$$[L_n, L_{-n}] = 2nL_0 + \frac{n^3-n}{12}.$$

So we get a Vir-module with $C=1$.

We obtain

<u>Theorem</u>. F_μ is a Vir-module of central charge $\underline{1}$.