Lecture 3.

Proof of the theorem

Let $(,)_{\lambda,n}: U(n_-)[-n] \otimes U(n_+)[n] \to \mathbb{C}$ be given by the formula

$$(a, b)_{\lambda,n} = (a v_\lambda^+, b v_{-\lambda}^-) = (S(b) a v_\lambda^+, v_{-\lambda}^-),$$

$a \in U(n_-)[-n]$, $b \in U(n_+)[n]$.

From this formula it is clear that $(,)_{\lambda,n}$ is polynomial with respect to λ (we commute factors of b past the factors of a, and when we get elements of $\mathfrak{g}_0$, we move them to the right, and when they hit v_λ^+, they produce linear functions in λ). So one can define the determinant of $(,)_{\lambda,n}$, which is well defined up to a constant factor (if we change bases in $U(n_\pm)[\pm n]$, the determinant gets multiplied by a constant). So the theorem follows from the following proposition.

Define the bilinear form $(,)^0_{\lambda,n}: Sn_-[-n] \otimes Sn_+[n] \to \mathbb{C}$ obtained by restriction of the form

$$\bigoplus_{k\geq 0} \frac{\lambda([\,,\,])_n^{\otimes k}}{k!} : Tn_-[-n] \otimes Tn_+[n] \to \mathbb{C}$$

to symmetric powers. Clearly, $\det(\,)^0_{\lambda,n}$ is a homogeneous polynomial, which is not identically zero, since $\lambda([\,,\,])$ is genesically nondegenerate.

<u>Proposition</u>. The leading term of $\det(\,,\,)_{n,\lambda}$ is $\det(\,,\,)^0_{n,\lambda}$.

<u>Proof</u>. Consider the Lie algebra $\mathfrak{g}[\varepsilon]$ and the subalgebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}[\varepsilon]$ generated by $\varepsilon\mathfrak{g}[i]$ for $i \neq 0$ and $\varepsilon^2\mathfrak{g}[0]$. Then $\tilde{\mathfrak{g}}/(\varepsilon-\alpha) \cong \mathfrak{g}$ $\forall \alpha \neq 0$, and $\tilde{\mathfrak{g}}/(\varepsilon) \cong \bar{\mathfrak{g}}$, where $\bar{\mathfrak{g}} = \mathfrak{g}$, but the bracket is 0, except $\mathfrak{g}[i] \otimes \mathfrak{g}[-i] \to \mathfrak{g}[0]$, which is the usual bracket (this is a kind of Heisenberg algebra). So we see that we can degenerate $\mathfrak{g}$ to a Heisenberg algebra (or represent $\mathfrak{g}$ as a deformation of a Heisenberg algebra).

Note that $(\, , \,)^{(\mathfrak{g})}_{\lambda,n} = (\, , \,)^{\circ}_{\lambda,n}(\mathfrak{g})$,
and $(\, , \,)^{(\tilde{\mathfrak{g}})}_{\lambda,n} = \varepsilon^{2l_n}(\, , \,)_{\frac{\lambda}{\varepsilon^2}, n}(\mathfrak{g})$

where l_n is the degree of this polynomial. This implies the proposition.

Now we can develop the basic representation theory of $\mathfrak{g}$.

Let $J_\lambda^\pm$ be the kernel of $(\, , \,)$ on $M_\lambda^\pm$. Clearly, $J_\lambda^\pm$ is a graded submodule, so one can define the graded $\mathfrak{g}$-module $L_\lambda^\pm = M_\lambda^\pm / J^\pm$. The form $(\, , \,)$ descends to a nondegenerate form $(\, , \,)_\lambda : L_\lambda^+ \otimes L_\lambda^- \to \mathbb{C}$.

<u>Theorem.</u> (I) $L_\lambda^\pm$ is an irreducible module.

(II) $J_\lambda^\pm$ is the maximal proper graded submodule of $M_\lambda^\pm$ (contains all other graded proper submodules)

(III) If $\exists L \in \mathfrak{g}_0$ such that $\mathrm{ad}\, L = n$ on $\mathfrak{g}[n]$ then $J_\lambda^\pm$ is the maximal proper submodule of $M_\lambda^\pm$ (contains all other proper submodules)

<u>Proof.</u> (I). Let us show that L_λ^+ is irreducible (the proof for L_λ^- is the same). Assume the contrary. Let $W \subset L_\lambda^+$ be

a proper submodule. Recall that L^+_λ is graded by negative integers. Pick $w \in W$, $w \neq 0$, such that w has ~~smallest~~ degree $-m \leq 0$ with smallest m (i.e. $w = \sum_{i=0}^{-m} w_i$, w_i has degree i). (It's clear that $m \neq 0$)

Then $aw = 0$ for all $a \in \mathfrak{g}[j]$, $j > 0$, so $aw_{-m} = 0$. Now consider (w_{-m}, bu), $b \in \mathfrak{g}[j]$, $j > 0$, $u \in L^-_{-\lambda}$. This equals $(-bw_{-m}, u) = 0$. But any vector in $L^-_{-\lambda}$ in degree m is a linear combination of vectors of the form bu. So $w_{-m} \in \mathrm{Ker}(\,,\,)$, $\Rightarrow\Leftarrow$.

(II) If K is another graded proper submodule then $K + J^\pm_\lambda$ is one as well, so its image in $L^\pm_\lambda$ is a proper graded submodule, hence 0. Thus, $K + J^\pm_\lambda = J^\pm_\lambda$, so $K \subset J^\pm_\lambda$.

(III). If $\exists L \in \mathfrak{g}_0$ as stated then grading is internal, so all submodules are graded.

<u>Remark</u>. If $\mathfrak{g}$ is the Heisenberg algebra then there are proper submodules which are not graded.

Theorem. For Weil generic λ, $M_\lambda^\pm$ are irreducible (Weil generic = away from countable union of hypersurfaces)

Proof. This follows from the determinant theorem and the previous theorem.

Now let us define category $\mathcal{O}^+$ of graded $\mathfrak{g}$-modules.

Definition. $M \in \mathcal{O}^+$ if M is a $\mathbb{C}$-graded $\mathfrak{g}$-module, such that all degrees lie in a half-plane $\mathrm{Re}(z) < a$, and fall into finitely many arithmetic progressions, and if $\forall d$ $M[d]$ is finite dimensional. Similarly one defines $\mathcal{O}^-$.

For example, $M_\lambda^\pm$ and $L_\lambda^\pm$ are in $\mathcal{O}^\pm$.

Proposition. $L_\lambda^\pm$ are the only irreducible objects in $\mathcal{O}^\pm$, and they are pairwise nonisomorphic.

Proof. They are pairwise nonisomorphic since $L_\lambda^\pm$ has a unique vector $v_\lambda^\pm$ killed by $\mathfrak{g}[j]$, $j>0$, and such that $av_\lambda^\pm = \lambda(a) v_\lambda^\pm$, $a \in \mathfrak{g}[0]$. Also, if $M \in \mathcal{O}$, let d be a maximal degree in M (in terms of real part). ~~Then~~ Let v be a common eigenvector of $\mathfrak{h} = \mathfrak{g}[0]$ in $M[d]$. Then $av = \lambda(a)v$ for $\lambda \in \mathfrak{h}$, and so we have a morphism

$$M_\lambda^+ \to M, \quad v_\lambda^+ \mapsto v.$$

If M is irreducible, this map is surjective, and kernel is a graded submodule, so $\mathrm{Ker} = J_\lambda^+$ and $M \cong L_\lambda^+$.

<u>Definition</u> The character of $M \in \mathcal{O}^+$ is

$$\chi_M(q,a) = \sum_{d \in \mathbb{C}} q^{-d} \, \mathrm{Tr}|_{M[d]}(e^a), \quad a \in \mathfrak{h}.$$

Note that if $\exists L \in \mathfrak{g}[0]$ acting by degree, then for an irreducible M,

$$\chi_M(q,a) = q^\alpha \sum_d \mathrm{Tr}|_{M[d]}(e^{a + L \log q}),$$

so the character up to factor is defined by $\mathrm{Tr}_M(e^a)$ (which is convergent in an appropriate sense).

Example. If we put $\mathbb{Z}_-$-grading on M_λ^+:

$$\chi_{M_\lambda^+}(q,a) = \prod_{j>0} \frac{1}{\det_{\mathfrak{g}[-j]}\left(1-q^j e^{ad(a)}\right)}.$$

Def. A Highest weight module with highest weight λ is any graded quotient of M_λ^+. Similarly, a lowest weight module is any graded quotient of M_λ^-. It follows from the above that any highest weight module with h.w. λ carries a pairing with $M_{-\lambda}^-$.