Book Lectures on Repn thry & KZ equations. (by Etingof & others)

Interpretation of solutions of KZ eqns in terms of $\hat{\mathfrak{g}}$.

$L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$, $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$ (K central)

$\tilde{\mathfrak{g}} = \mathbb{C}d \ltimes \hat{\mathfrak{g}}$ (different not$^{\underline{n}}$ than before).

Assume V_i fin. dim for convenience.

1) Weyl modules defined for $\lambda \in P_T$, $k \in \mathbb{C}$

Let L_λ be the irred repn of $\mathfrak{g}$ of h. wt. λ. Regard L_λ as a $\mathfrak{g}[t] \oplus \mathbb{C}K$ - module by setting the action of $t\mathfrak{g}[t]$ to be 0 & K acts by k.

Defn The Weyl module is

$$V_{\lambda,k} := \mathrm{Ind}_{\mathfrak{g}[t]\oplus \mathbb{C}K}^{\hat{\mathfrak{g}}} L_\lambda$$

$$= U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus \mathbb{C}K)} L_\lambda .$$

As a vector space this is $U(t^{-1}\mathfrak{g}[t^{-1}]) \otimes L_\lambda$.

It looks like:

0 ———— L_λ

-1 ———— $t^{-1}\mathfrak{g} \otimes L_\lambda$

-2 ———— $t^{-2} S^2(\mathfrak{g}) \otimes L_\lambda \oplus t^{-2}\mathfrak{g} \otimes L_\lambda$

$\vdots$

If $k \neq -h^\vee$ then $V_{\lambda,k}$ can be extended to a $\tilde{\mathfrak{g}}$-module by setting $d = -L_0$ where L_0 is the operator constructed by Sugawara,

$$L_0 = \frac{1}{2(k+h^\vee)} \sum_{i,\, a \in B} :at^i \cdot at^{-i}:$$

B an ON basis of $\mathfrak{g}$.

2) Laurent polynomial modules.

If V is a $\mathfrak{g}$-module then $V[z,z^{-1}]$ is a $L\mathfrak{g}$-module in the obvious way ($t \mapsto z$). Can view it as a $\hat{\mathfrak{g}}$-module by putting $K=0$. Can extend to $\tilde{\mathfrak{g}}$-module by

$$d(vz^n) = n v z^n .$$

In fact, for any $\Delta \in \mathbb{C}$ we can define

$$d(vz^n) = (n-\Delta) v z^n$$

In this case we can rename vz^n as $vz^{n-\Delta}$ and call the extended module

$$z^{-\Delta} V[z,z^{-1}] .$$

<u>Lemma</u> If $k \notin \mathbb{Q}$ then $V_{\lambda,k}$ is irred.

<u>Pf</u> Assume $V_{\lambda,k}$ has a singular vector w in some degree $\ell > 0$. The Casimir is

$$C = L_0 + \partial$$

∂ — usual grading operator.

$$C|_{V_{\lambda,k}} = \frac{(\lambda, \lambda+2\rho)}{2(K+h^\vee)}$$

ρ for $\mathfrak{g}$

Suppose $w \in L_\mu$, $\mu \in P_+$ WLOG.

Then $$C|_w = \frac{(\mu, \mu+2\rho)}{2(k+h^\vee)} - \ell$$

Thus $$(\underset{\mathbb{Z}}{\underset{\Uparrow}{\lambda}}, \lambda+2\rho) = (\mu, \underset{\mathbb{Z}}{\underset{\Uparrow}{\mu}}+2\rho) - 2\ell(k+h^\vee)$$

which cannot happen if $k \notin \mathbb{Q}$. $\boxtimes$

Corollary $V^*_{\lambda,k}$ is $\mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}[t^{-1}] \oplus \mathbb{C}K)} L^*_\lambda$

$$t^{-1}\mathfrak{g}[t^{-1}]|_{L^*_\lambda} = 0, \quad K|_{L^*_\lambda} = -k.$$

Pf Let $M = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}[t^{-1}] \oplus \mathbb{C}K)} L^*_\lambda$

Then $\exists!$ a homom' $\phi : M \longrightarrow V^*_{\lambda,k}$,

$\phi|_{\text{lowest degree}} = id_{L^*_\lambda}$. Since $V^*_{\lambda,k}$ is irred, this is surjective. But the characters are equal so it is an isomorphism. $\boxtimes$

<u>Prop</u>: If $k \notin \mathbb{Q}$ then

$$\mathrm{Hom}_{\tilde{\mathfrak{g}}}(V_{\lambda,k} \otimes V^*_{\nu,k},\ z^{-\Delta} V[z,z^{-1}]) \cong \mathrm{Hom}_{\mathfrak{g}}(L_\lambda \otimes L^*_\nu,\ V)$$

if $\Delta = \Delta(\lambda) - \Delta(\nu)$.

Pf Frobenius reciprocity.

$$\mathrm{Hom}_{\tilde{\mathfrak{g}}}(V_{\lambda,k} \otimes V^*_{\nu,k},\ z^{-\Delta} V[z,z^{-1}])$$

$$= \mathrm{Hom}_{\mathfrak{g}[t^{-1}] \oplus \mathbb{C}k \oplus \mathbb{C}d}(V_{\lambda,k} \otimes L^*_\nu,\ z^{-\Delta} V[z,z^{-1}])$$

$$= \mathrm{Hom}_{\mathfrak{g} \oplus \mathbb{C}d}(L_\lambda \otimes L^*_\nu,\ z^{-\Delta} V[z,z^{-1}]) \qquad (*)$$

$d|_{L_\lambda \otimes L^*_\nu} = -\Delta(\lambda) + \Delta(\nu) = -\Delta$, so need to isolate the piece of $z^{-\Delta} V[z,z^{-1}]$ on which d acts by Δ, this is $z^{-\Delta} V$.

So $(*) \cong \mathrm{Hom}_{\mathfrak{g}}(L_\lambda \otimes L^*_\nu,\ V)$.

We will view elements of $\mathrm{Hom}_{\tilde{\mathfrak{g}}}(V_{\lambda,k} \otimes V^*_{\nu,k},\ z^{-\Delta} V[z,z^{-1}])$ as

$$\phi(z) : V_{\lambda,k} \longrightarrow V_{\nu,k} \,\hat{\otimes}\, z^{-\Delta} V[z,z^{-1}].$$

$$\Big(V_{\nu,k} \,\hat{\otimes}\, E := \prod_i V^{(i)}_{\nu,k} \otimes E\Big).$$

degree i part.

$$\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n-\Delta}$$ where $\phi_n : V_{\lambda,k} \to V_{\nu,k} \otimes V$ a $\mathfrak{g}$-homom.

$$\phi(z) w \in V_{\nu,k} \otimes V((z)).$$

(Tsuchiya - Kanie).

λ_j, $j = 0, \dots, N$.
ν_j, $j = 1, \dots N$.

$$\phi^j(z_j) : V_{\lambda_j, k} \longrightarrow V_{\lambda_{j-1}, k} \hat{\otimes} z^{-\Delta_j} L_{\mu_j}[z_j, z_j^{-1}].$$

$$\Delta_j = \Delta(\lambda_j) - \Delta(\lambda_{j-1}).$$

indices increase

$$V_{\lambda_N, k} \xrightarrow[\phi^N(z_N)]{} V_{\lambda_{N-1}, k} \hat{\otimes} z_N^{-\Delta_N} L_{\mu_N}[z_N, z_N^{-1}]$$

$$\xrightarrow[\phi^{N-1}(z_{N-1})]{} V_{\lambda_{N-2}, k} \hat{\otimes} z^{-\Delta_N} L_{\mu_{N-1}}[z_{N-1}, z_{N-1}^{-1}] \hat{\otimes} z^{-\Delta_{N-1}} L_{\mu_N}[z_N, z_N^{-1}]$$

$$\longrightarrow \dots \xrightarrow[\phi^1(z_1)]{} V_{\lambda_0, k} \hat{\otimes} z^{-\Delta_N} L_{\mu_1}[z_1, z_1^{-1}] \hat{\otimes} \dots \hat{\otimes} z^{-\Delta_1} L_{\mu_N}[z_N, z_N^{-1}].$$

Composition is $\phi^1(z_1) \dots \phi^N(z_N)$
If u is a homog'. vector in $V_{\lambda_N, k}$ and u' is a homog' vector in $V^*_{\lambda_0, k}$ then

$$\langle u' | \phi^1(z_1) \dots \phi^N(z_N) u \rangle$$

$$\in L_{\mu_1} \otimes \dots \otimes L_{\mu_N} z_1^{\alpha_1} \dots z_N^{\alpha_N} \mathbb{C}\left[\left[\frac{z_2}{z_1}, \frac{z_3}{z_1}, \dots, \frac{z_N}{z_{N-1}}\right]\right]$$

(exercise).

So if $u_0 \in L^*_{\lambda_0}$, $u_i \in L^*_{\mu_i}$, $u_{N+1} \in L_{\lambda_N}$ then one can consider

$$\psi_{u_0; u_1, \dots, u_N, u_{N+1}}(z_1, \dots, z_N)$$

$$= \langle u_0 \otimes \dots \otimes u_N \,|\, \phi^1(z_1) \dots \phi^N(z_N) u_{N+1} \rangle$$

$$\in z_1^{-\Delta_1} \dots z_N^{-\Delta_N} \mathbb{C}[[\tfrac{z_2}{z_1}, \dots, \tfrac{z_N}{z_{N-1}}]].$$

Let $\psi(z_1, \dots z_N) \in (L_{\lambda_0} \otimes L_{\mu_1} \otimes \dots \otimes L_{\mu_N} \otimes L^*_{\lambda_{N+1}})^{\mathfrak{g}}$ be the generating function.

$$\langle u_0 \otimes \dots \otimes u_N \,|\, \phi^1(z_1) \dots \phi^N(z_N) u_{N+1} \rangle = \langle u_0 \otimes \dots \otimes u_N \otimes u_{N+1} \,|\, \psi(z_1 \dots z_N) \rangle$$

Let $\hat{\psi}(z_1 \dots z_N) := \left(\prod_{i=1}^{N} z^{-\Delta(\mu_i)}\right) \psi(z_1, \dots z_N)$. (Renormalise)

<u>Theorem</u> $\hat{\psi}(z_1 \dots z_N)$ satisfies the KZ equation

$$\varkappa \frac{\partial \hat{\psi}}{\partial z_i} = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \hat{\psi} \quad (i = 1 \dots N)$$

with $z_{N+1} = 0$.

Such solutions <u>span</u> the space of all solutions, in the sense that $\langle u_0 | \psi \rangle$ for various λ_0, u_0 span the space of sol$^{\underline{n}}$ to KZ in $L_{\mu_1} \otimes \dots \otimes L_{\mu_N} \otimes L^*_{\lambda_N}$.

<u>Cor</u> $\left(\prod z_i^{\Delta_i + \Delta(\mu_i)}\right) \hat{\psi}(z_1 \dots z_N) = \left(\prod z_i^{\Delta_i}\right) \psi(z_1, \dots z_N)$ converges for $\left\{ \left|\frac{z_i}{z_{i-1}}\right| < 1 \ \ \forall i \right\}$.

<u>Pf</u> General theory of compatible systems of ODEs. —

Suppose $A_i(\xi_1,\dots\xi_m)$ are holom' fcts in $\{|\xi_j|<1\}$ with values in $\mathrm{Mat}_n(\mathbb{C})$ and consider the diff. eqns

$$\xi_i \frac{\partial F}{\partial \xi_i} = A_i(\xi)F \qquad \left(F(\xi_1,\dots\xi_m)\in\mathbb{C}^n\right).$$

Assume the eqns are consistent, i.e.,

$$\left[\xi_i\frac{\partial}{\partial\xi_i}-A_i,\ \xi_j\frac{\partial}{\partial\xi_j}-A_j\right]=0$$

(zero curvature equations)

$$\left(\Rightarrow \xi_i\partial_{\xi_i}A_j-\xi_j\partial_{\xi_j}A_i=[A_i,A_j]\right).$$

Assume that $\{A_i(0)\}$ have a common eigenbasis $\{v_k\}$ $k=1\dots n$ s.t. $A_i(0)v_k=\lambda_{ik}v_k$ and $\forall\, k,l$

$$(\lambda_{1k}-\lambda_{1l},\lambda_{2k}-\lambda_{2l},\dots\lambda_{mk}-\lambda_{ml})\notin\mathbb{Z}^m\setminus\{0\}.$$

Then 1 $\exists!$ matrix solution

$$\mathcal{F}(\xi_1,\dots\xi_m)=\mathcal{F}_0(\xi_1,\dots\xi_m)\,\xi_1^{A_1(0)}\cdots\xi_m^{A_m(0)}.$$

$\mathcal{F}_0$ is holom' in $|\xi_j|<1$ and $\mathcal{F}_0(0,0,\dots 0)=1$ ("Fundamental sol$^{\underline{n}}$")

2 Any sol'n in $\mathbb{C}^n$ in a simply conn' region contained in $\{0<|\xi_i|<1\}$ is of the form $\mathcal{F}(\xi_1,\dots\xi_m)v$ for a unique vector $v\in\mathbb{C}^n$.

3 If $f(\xi_1, \ldots, \xi_m) = \xi_1^{r_1} \ldots \xi_m^{r_m} \mathbb{C}^n[[\xi_1, \ldots, \xi_m]]$ is a formal solution then it is convergent and of the form (2).

Lecture 3
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The theorem (in 1-variable case) is

Thm Let $A(z) = A_0 + A_1 z + \dots$ be a holom fct for $|z| < 1$ with values in $Mat_N(\mathbb{C})$ and assume that for any two distinct eigenvalues of A_0 (λ & μ), $\lambda - \mu \notin \mathbb{Z}$.
Then the ODE
$$z \frac{dF}{dz} = A(z) F$$
has a matrix-valued solution of the form
$$F(z) = (1 + B_1 z + B_2 z^2 + \dots) z^{A_0}$$
st. the series $1 + B_1 z + \dots$ converges for $|z| < 1$.

Any vector-valued sol'n of the eqn in a simply-conn region of $\mathbb{C}^*$ has the form $F(z) v$ ($v \in \mathbb{C}^N$). Moreover any formal sol'n is actually convergent.

Note: $z^{A_0} = e^{A_0 \log z}$, multivalued function

Pf Plug in $F(z)$ & look @ coeff. of $z^n \cdot z^{A_0}$

$$\Big(\sum_{n\geq 1} n B_n z^n\Big) z^{A_0} + \Big(1 + \sum_{n \geq 1} B_n z^n\Big) A_0 z^{A_0}$$
$$= (A_0 + A_1 z + \dots)(1 + B_1 z + B_2 z^2 + \dots) z^{A_0}.$$
$$\Rightarrow n B_n + B_n A_0 = A_0 B_n + E_n$$
$$\Rightarrow (n - \mathrm{ad} A_0) B_n = E_n.$$
expr. involving A_i & B_j with $j < n$.

Since A_0 has no two e-vals differing by elt of $\mathbb{Z} \setminus \{0\}$. Hence $n - \mathrm{ad} A_0$ is invertible.

$\Rightarrow B_n = (n - \mathrm{ad}\, A_0)^{-1} E_n$

Moreover an explicit calculation shows

$E_n = A_1 B_{n-1} + \dots + A_{n-1} B_1 + A_n$

Hence for any $C > 1$,

$\frac{|B_n|}{C^n} \to 0$ as $n \to \infty$ $\Rightarrow 1 + \sum z^n B_n$ converges for $|z| < 1$.

(use $\frac{|A_n|}{C^n} \to 0$ as $n \to \infty$).

Columns of $F(z)$ are lin. indep (on any simp. conn region) $\Rightarrow$ all vec-valued sol'ns are of the form $F(z)v$. $\boxtimes$

In the general case $z_i \frac{\partial}{\partial z_i} F = A_i(z) F$ on $\mathbb{C}^r$.

Pick a vector $\vec{w} = (w_1, \dots w_r)$, $\vec{z} = t\vec{w}$, $t \in \mathbb{R}$

get $t \frac{\partial}{\partial t} F(tw_1, \dots, tw_r) = \dots$

Solve to get solution along the ray $\mathbb{C}\vec{w}$.

We get sol'ns $\forall$ such rays, the resulting function on $\mathbb{C}^r$ obeys the ODE if

$[z_i \partial_i - A_i, z_j \partial_j - A_j] = 0 \quad \forall i, j.$ $\boxtimes$

Recall $\Phi^i(z) : V_{\lambda_i, k} \longrightarrow V_{\lambda_{i-1}, k} \hat{\otimes} z^{-\Delta_i} L_{\mu_i}[z, z^{-1}]$

$\left(\Delta_i := \Delta(\lambda_i) - \Delta(\lambda_{i-1}), \quad \Delta(\lambda) = \frac{(\lambda, \lambda + 2\rho)}{2\varkappa}, \quad \varkappa = k + h^\vee\right)$

$\hat{\phi}^i(z) = z^{-\Delta(\mu_i)} \phi^i(z), \quad \forall u \in L^*_{\mu_i}$

$\hat{\Phi}^i_u(z) = \langle 1 \otimes u \,|\, \hat{\phi}^i(z) \rangle : V_{\lambda_i, k} \longrightarrow z^{-\Delta_i - \Delta(\mu_i)} V_{\lambda_{i-1}, k}((z))$

$$\langle u_0 | \hat{\Phi}^1_{u_1}(z_1) \dots \hat{\Phi}^N_{u_N}(z_N) u_{N+1} \rangle \in \prod_i z_i^{-\Delta_i - \Delta(\mu_i)} \mathbb{C}[[\tfrac{z_2}{z_1}, \dots \tfrac{z_N}{z_{N-1}}]]$$

$u_0 \in L^*_{\lambda_0}$, $u_i \in L^*_{\mu_i}$ $(i = 1 \dots N)$, $u_{N+1} \in L_{\lambda_N}$.

Call this function $\psi_{u_0, \dots u_{N+1}}(z_1, \dots z_N)$ for brevity.

Define $\psi(z_1 \dots z_N) \in L_{\lambda_0} \otimes L_{\mu_1} \otimes \dots \otimes L_{\mu_N} \otimes L^*_{\lambda_N}$ by

$$\langle u_0 \otimes u_1 \otimes \dots \otimes u_N \otimes u_{N+1} | \psi \rangle = \psi_{u_0, \dots, u_{N+1}}$$

This is called a correlation function of intertwining operators.

<u>Thm</u> $\psi(z_1 \dots z_N)$ satisfies the KZ eqns,

$$\varkappa \frac{\partial \psi}{\partial z_i} = \sum_{\substack{j \neq i \\ 1 \le i,j \le N}} \frac{\Omega_{ij}}{z_i - z_j} \psi + \frac{\Omega_{i,N+1}}{z_i} \psi$$

in $N+1$ variables $z_1, \dots z_{N+1}$ with $z_{N+1} \equiv 0$.

<u>Cor</u> The series $\psi(z_1 \dots z_N)$ converges for $|z_1| > |z_2| > \dots > |z_N| > 0$.

Pf of Cor: Can write KZ as

$$\varkappa z_i \frac{\partial \psi_i}{\partial z_i} = \sum_{j \neq i} \frac{\Omega_{ij}}{1 - \frac{z_j}{z_i}} \psi + \Omega_{i,N+1} \psi.$$

Conjugate by some z^B (B a matrix) to get a system for which $\sum z_i \frac{\partial}{\partial z_i} F = 0$

Pass to variables $\zeta_i = \frac{z_{i+1}}{z_i}$.

$$\varkappa\, \zeta_i \frac{\partial F}{\partial \zeta_i} = (A_{0i} + \dots) F.$$

The A_{0i} have differences of eigenvalues in $\mathbb{Q}$ since the e-vals themselves are in $\mathbb{Q}$.
$\therefore$ after dividing by $\varkappa$ (which we assume $\notin \mathbb{Q}$) the differences $\notin \mathbb{Q}$.

So the theorem on ODEs applies & we get the result. $\boxtimes$

<u>Pf of thm</u>: We introduce currents

$$J_a(z) = \sum_{n \in \mathbb{Z}} (at^n) z^{-n-1}. \quad (a \in \mathfrak{g}).$$

$$= J_a^+ + J_a^-$$

$\sum_{n<0}$, holomorphic part. $\qquad$ $+\sum_{n\geq 0}$, singular part.

If $v \in z^\gamma V[z, z^{-1}]$ then

$$J_a^+(\zeta) v = \frac{av}{z-\zeta} := \frac{av}{z} + \frac{av}{z^2}\zeta + \frac{av}{z^3}\zeta^2 + \dots \left(\text{+ve powers of } \zeta\right)$$

$$J_a^-(\zeta) v = -\frac{av}{z-\zeta} \quad (\text{+ve powers of } z)$$

So $J_a(\zeta) v = \frac{av}{z} \delta(\zeta/z)$

Lemma $[J_a^{\pm}(\zeta), \phi_u(z)] = \frac{1}{z-\zeta}\phi_{au}(z)$.

Pf $[a_{(n)}, \phi_u(z)] = z^n \phi_{au}(z)$

<u>Main lemma</u> $\varkappa \frac{d\hat{\phi}_u(z)}{dz} = \sum_{a\in B} :J_a(z)\hat{\phi}_{au}(z):$

where $:J_a(z)\hat{\phi}_u(z): = J_a^+(z)\hat{\phi}_u(z) - \hat{\phi}_u(z)J_a^-(z)$.
as usual.

Pf $z\frac{d\phi_u}{dz} = [\phi_u(z), d] \Rightarrow z\frac{d\hat{\phi}_u}{dz} = [\hat{\phi}_u, d] - \Delta(\mu)\hat{\phi}_u$

(since $\hat{\phi} = z^{-\Delta(\mu)}\phi$).

Now recall that d acts on $V_{\lambda,k}$ by $-L_0$. Thus $z\frac{d\hat{\phi}_u}{dz} = [L_0, \hat{\phi}_u] - \Delta(\mu)\hat{\phi}_u$.

Now use Sugawara construction:

$$L_0 = \frac{1}{2\varkappa}\sum_{a\in B}\left(\sum_{n>0} a_{(-n)}a_{(n)} + \sum_{n\le 0} a_{(n)}a_{(-n)}\right)$$

$$\Rightarrow z\frac{d\hat{\phi}_u}{dz} = \frac{1}{2\varkappa}\sum_{a\in B}\Big(\sum_{n>0}(z^n a_{(-n)}\hat{\phi}_{au} + z^{-n}\hat{\phi}_{au}a_{(n)}) + \sum_{n\le 0}(z^{-n}a_{(n)}\hat{\phi}_{au} + z^n\hat{\phi}_{au}a_{(-n)})\Big) - \Delta(\mu)\hat{\phi}_u.$$

$$= \frac{1}{\varkappa} z\sum_{a\in B}\left[J_a^+(z)\hat{\phi}_{au}(z) - \hat{\phi}_{au}(z)J_a^-(z)\right] + \sum_{a\in B}\frac{1}{2\varkappa}[a_{(0)}, \hat{\phi}_{au}] - \Delta(\mu)\hat{\phi}_u.$$

but $\frac{1}{2\varkappa}\sum_{a\in B}[a_{(0)}, \hat{\phi}_{au}] = \frac{1}{2\varkappa}\hat{\phi}_{\sum_{a\in B} a^2 u} = \frac{1}{2\varkappa}(\mu,\mu+2\rho)\hat{\phi}_u$

$$= \Delta(\mu)\hat{\phi}_u . \boxtimes$$

& we're done almost.

$$\varkappa\frac{\partial\psi_{u_0\ldots u_{N+1}}}{\partial z_i} = \sum_{a\in B}\langle u_0|\hat{\phi}^1_{u_1}(z_1)\ldots\big(J^+_a(z_i)\hat{\phi}^i_{u_i}(z_i) - \hat{\phi}^i_{u_i}(z_i)J^-_a(z_i)\big)\ldots\hat{\phi}^N_{u_n}(z_N)u_{N+1}\rangle$$

commute J_a^+ to the far left & J_a^- to the far right.

$$= \sum_{j<i}\frac{\langle u_0|\ldots\hat{\phi}^j_{au_j}(z_j)\ldots\hat{\phi}^i_{au_i}(z_i)\ldots\rangle}{z_i - z_j}$$

$$+ \sum_{j>i}\frac{\langle u_0|\ldots\hat{\phi}^i_{au_i}(z_i)\ldots\hat{\phi}^j_{au_j}(z_j)\ldots\rangle}{z_j - z_i}.$$

might be some wrong signs here...

$$\frac{1}{z_i}\langle u_0|\ldots\hat{\phi}^i_{au_i}(z_i)\ldots a u_{N+1}\rangle$$

$$\Longrightarrow \varkappa\frac{\partial\psi}{\partial z_i} = \sum_{j\neq i}\frac{\Omega_{ij}}{z_i - z_j}\psi + \frac{\Omega_{i,N+1}}{z_i}\psi. \quad \boxtimes .$$

Solutions of the KZ equations.

Lecture 4
5/4/2010

<u>Prop</u> Correlation fcts $\psi(z_1, \ldots z_N)$ for various choices of $\phi^i(z_i)$ span the space of solutions of KZ.

<u>Pf</u> $\phi^i(z) : V_{\lambda_i, k} \longrightarrow V_{\lambda_{i-1}, k} \otimes z^{-\Delta_i} L_{\mu_i}[z, z^{-1}]$.

$\{\phi^i\} \cong \mathrm{Hom}_{\mathfrak{g}}(L_{\lambda_i}, L_{\lambda_{i-1}} \otimes L_{\mu_i})$

Given $u_0 \in L^*_{\lambda_0}$ and $\phi^1 \ldots \phi^N$ we define a sol'n
$\psi_{u_0}(z_1 \ldots z_N) = \langle u_0 | \phi^1_?(z_1) \ldots \phi^N_?(z_N) ? \rangle \in L_{\mu_1} \otimes \ldots \otimes L_{\mu_n} \otimes L^*_{\lambda_n}$

of KZ eqns with variables $z_1, \ldots z_N$, & $z_{N+1} \equiv 0$.
So we get a space of solutions parameterised by

$$\bigoplus_{\lambda_0, \ldots \lambda_{N-1}} \left(L^*_{\lambda_0} \otimes \mathrm{Hom}_{\mathfrak{g}}(L_{\lambda_1}, L_{\lambda_0} \otimes L_{\mu_1}) \otimes \ldots \otimes \mathrm{Hom}_{\mathfrak{g}}(L_{\lambda_N}, L_{\lambda_{N-1}} \otimes L_{\mu_N}) \right)$$

$$= \bigoplus_{\lambda_0} L^*_{\lambda_0} \otimes \bigoplus_{\lambda_1 \ldots \lambda_{N-1}} (\text{" " "})$$

$$= \bigoplus_{\lambda_0} L^*_{\lambda_0} \otimes \mathrm{Hom}(L_{\lambda_N}, L_{\lambda_0} \otimes L_{\mu_1} \otimes \cdots \otimes L_{\mu_N})$$

$$= \bigoplus_{\lambda_0} L^*_{\lambda_0} \otimes \mathrm{Hom}(L^*_{\lambda_0}, L_{\mu_1} \otimes \ldots \otimes L_{\mu_N} \otimes L^*_{\lambda_N})$$

$$= L_{\mu_1} \otimes L_{\mu_2} \otimes \ldots \otimes L_{\mu_N} \otimes L^*_{\lambda_N}$$

which is exactly as many as we want.

So for each simp. conn. region D in which $|z_1|<|z_2|<\dots<|z_N|$ we get a holom'. matrix solution of the KZ eqn with values in
$$\mathrm{End}(\underbrace{L_{\mu_1}\otimes\dots\otimes L_{\mu_N}\otimes L^*_{\lambda_N}}_{=Q}).$$
(b/c for any elt of Q we got a Q-valued soln of KZ).

By sending $z_{i+1}/z_i \to 0$ we see the fct is generically non-deg.
$\Rightarrow$ Corr. fcts span the space of soln to KZ.

Solutions

Let M_ν be a <u>lowest</u> wt. Verma module for $\mathfrak{g}$ with lowest wt ν.
$$\varkappa \frac{\partial F}{\partial z_i} = \sum \frac{\Omega_{ij}}{z_i - z_j} F$$
Let $\mu_1,\dots\mu_N$ be generic, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.
$$F(z_1,\dots z_N) \in (M_{-\mu_1}\otimes\dots\otimes M_{-\mu_N})^{\mathfrak{n}_-}.$$
$\uparrow$
determines all sol$\underline{\text{n}}$ since for generic $\{\mu_i\}$
$$M_{-\mu_1}\otimes\dots\otimes M_{-\mu_N} = \bigoplus_{\beta\in Q_+} M_{-(\mu_1+\dots+\mu_N)+\beta} \otimes \text{(PTO)}$$

$$(PTO) = \mathrm{Hom}(M_{-(\mu_1+\dots+\mu_N)+\beta}, M_{-\mu_1}\otimes\dots\otimes M_{-\mu_N})$$
$$\cong$$
$$(M_{-\mu_1}\otimes\dots\otimes M_{-\mu_N})^{n_-}[-\mu_1-\dots-\mu_N+\beta].$$

So all solns with values in $M_{-\mu_1}\otimes\dots\otimes M_{-\mu_N}$ are obtained from those in
$$(M_{-\mu_1}\otimes\dots\otimes M_{-\mu_N})^{n_-}.$$

So we'll look at solns with values in
$$(M_{-\mu_1}\otimes\dots\otimes M_{-\mu_N})^{n_-}[-(\mu_1+\dots+\mu_N)+\beta].$$

(Call these "solns of level β").

<u>1 Level 0</u> Let v_i be the lowest wt vector of $M_{-\mu_i}$ & $v = v_1\otimes\dots\otimes v_N$.
A level 0 soln is $\psi_0(z_1,\dots z_N)v$, it's the only one.

$\Omega_{ij}v = ?$ Well,

kills.

$$\Big(\sum_{a\in B} a\otimes a\Big)v_\ell\otimes v_j = \Big(\sum_i x_i\otimes x_i + \sum_{\alpha>0}(e_\alpha\otimes f_\alpha + f_\alpha\otimes e_\alpha)\Big)v_\ell\otimes v_j$$
$$= \sum x_i v_\ell\otimes x_i v_j = \Big(\sum_i \mu_\ell(x_i)\mu_j(x_i)\Big)(v_\ell\otimes v_j)$$
$$= \langle\mu_\ell,\mu_j\rangle(v_\ell\otimes v_j).$$

$\therefore$ <u>KZ eqn is</u> $\displaystyle \varkappa\frac{\partial\psi_0}{\partial z_i} = \sum_{j\neq i}\frac{\langle\mu_i,\mu_j\rangle}{z_i-z_j}\psi_0$

So $\displaystyle \psi_0 = C\prod_{i<j}(z_i-z_j)^{\langle\mu_i,\mu_j\rangle/\varkappa}.$

Now let $\mathfrak{g} = \mathfrak{sl}_2$, refer to level $m\alpha$ soln as level m.

So level 0 is $\psi_0(z) = \prod_{i<j} (z_i - z_j)^{\mu_i \mu_j / 2\varkappa}$.

We can do $N=2$ more generally,

$$\left.\begin{aligned} \varkappa \partial_{z_1} F &= \frac{\Omega}{z_1 - z_2} F \\ \varkappa \partial_{z_2} F &= \frac{\Omega}{z_2 - z_1} F \end{aligned}\right\} \Rightarrow F = (z_1 - z_2)^{\Omega/\varkappa}.$$

The 1st nontrivial case now is $\underline{N=3, \text{level } 1}$. The equation is now in the space

$$(M_{-\mu_1} \otimes M_{-\mu_2} \otimes M_{-\mu_3})^{\mathbb{C}\cdot f} [-\mu_1 - \mu_2 - \mu_3 + 2].$$

which is 2D with basis

$$w_1 = \mu_2 \, e v_1 \otimes v_2 \otimes v_3 - \mu_1 v_1 \otimes e v_2 \otimes v_3$$

$$\& \quad w_2 = \mu_3 \, v_1 \otimes e v_2 \otimes v_3 - \mu_2 v_1 \otimes v_2 \otimes e v_3.$$

Prop Any soln of level 1 is of the form

$$\psi(z_1, z_2, z_3) = (z_1 - z_3)^{(\Omega_{12} + \Omega_{23} + \Omega_{13})/\varkappa} f\left(\frac{z_1 - z_2}{z_1 - z_3}\right)$$ where

$f(z)$ is a sol'n of $\varkappa \frac{df}{dz} = \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right) f$.

which has reg. singular pts at $0, 1 \,\&\, \infty$.
$\Rightarrow$ it should reduce to the hypergeometric equation.

Remember their definition:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n \quad \left((q)_n := q(q+1)\dots(q+n-1)\right) $$

Converges for $|z|<1$.
Satisfies the hypergeometric equation,

$$ z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0. $$

<u>Abel's formula</u> says that the 2nd soln is

$$ z^{1-c} \, {}_2F_1(a-c+1, b-c+1; 2-c; z) $$

(except in degenerate cases).

Any 2×2 system with reg. sing. pts in 1 variable reduces to the HG equation.

<u>Thm</u> (Integral Repn)

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \, dt $$

$$ (\operatorname{Re} b > 0, \operatorname{Re}(c-b) > 0, |z|<1). $$

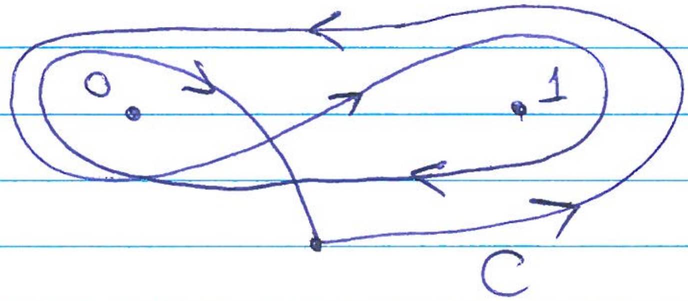
<u>Thm</u> $f(z)$ defined above

$$ = z^{[\mu_1\mu_2 - 2(\mu_1+\mu_2)]/2\varkappa} (1-z)^{\mu_2\mu_3/2\varkappa} \cdot $$

$$ \cdot \left(F(z) w_1 + z \frac{\varkappa}{\mu_3} F'(z) w_2 \right) $$

where F is any soln to the hypergeometric ODE.

The $\mathrm{Re}(b), \mathrm{Re}(c-b) > 0$ are annoying, let's $\int$ on a different path. But by multivaluedness we can't use any old loop. It needs to loop around 0 & 1, 0 times each.



will work. "Pochhammer loop".

$$\int_C t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

$$\longleftrightarrow \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt.$$

$$\int_C (\ldots) = \int_0^1 (\ldots) \cdot \underbrace{\left(1 - e^{2\pi i(c-b)} + e^{2\pi i c} - e^{2\pi i b}\right)}_{\begin{array}{c} \| \\ (1-e^{2\pi i b})(1-e^{2\pi i(c-b)}) \end{array}}.$$

So re-define $\int_0^1 (\ldots)$ as $\dfrac{1}{(1-e^{2\pi i b})(1-e^{2\pi i(c-b)})} \int_C (\ldots)$.

which works for arbitrary generic b, c.