Recall the denominator identity

$$\prod_{\alpha>0}(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \sum_{w\in W} \det(w)\, e^{w\rho-\rho} \qquad (*)$$

For $\mathfrak{g}=\hat{\mathfrak{g}}_0$, $\mathfrak{g}_0$ findim simple Lie Alg.

Positive roots $\quad \delta$ - basic imaginary root

1) $n\delta+\alpha \quad (\alpha\in\Delta)$, $n>0$ $\quad$ (Δ - roots of $\mathfrak{g}_0$)

2) $\alpha\in\Delta_+$

$\dim \mathfrak{g}[n\delta] = r = \dim \mathfrak{h}$ since $\mathfrak{g}[n\delta]=\mathfrak{h}t^n$.
$\dim \mathfrak{g}[\beta] = 1$ for other β.
Let $e^{-\delta}=:q$

$$\prod_{n=1}^{\infty}\left((1-q^n)^r \prod_{\alpha\in\Delta}(1-q^n e^{\alpha})\right)\cdot \prod_{\alpha>0}(1-e^{-\alpha})$$

a formal series in q, $z_i=e^{-\alpha_i}$ $\quad i=1\dots r$.

That was the LHS of $(*)$. The RHS

$W = W_0 \ltimes Q^\vee$; $Q^\vee$ the dual root lattice of $\mathfrak{g}_0$, spanned by $\alpha_i^\vee = h_i$. $(i=1,\dots,r)$.

Fact:
$\det(w_0, q) = \det(w_0)$.

$$RHS = \sum_{w_0 \in W_0} \sum_{y \in Q^\vee} \det(w_0) e^{w_0 y \rho - \rho}$$ ← quadratic in y.

So we get some kind of Θ function in $q, z_1, \dots z_n$.

After computation, the answer is

$$LHS = e^{-\rho_0} \sum_{\mu \in \frac{1}{2}Q^\vee} q^{(\mu + 2\rho_0, \mu)} \mathcal{J}_{\mu + \rho_0}.$$

where

$$\mathcal{J}_\lambda = \sum_{w_0 \in W_0} \det(w_0) e^{w_0 \lambda}$$

What happens if all $z_i = e^{-\alpha_i} = 1$?
On LHS the 2nd factor $\prod(1-e^{-\alpha})$ becomes 0 so we first divide it out and then take the limit.

LHS becomes $\prod_{n=1}^{\infty}(1-q^n)^{\dim \mathfrak{g}} = \varphi(q)^{\dim \mathfrak{g}}$.

RHS becomes

$$\sum_{\mu \in \frac{1}{2}Q^\vee} q^{(\mu + 2\rho_0, \mu)} \prod_{\alpha > 0} \frac{(\alpha, \mu + \rho_0)}{(\alpha, \rho_0)}$$

Eg// $\widehat{sl}_2$ gives the Jacobi triple product identity

$$\prod_{n}(1-q^n)(1-q^{n-1}z)(1-q^n z^{-1}) = \dots$$

$-176 \quad \sum_{m\in\mathbb{Z}} q^{m(2m+1)}(4m+1)$

to get $\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}$.
which was originally found by Gauss.

Knizhnik-Zamolodchikov equations.

$\mathfrak{g}$ fin. dim simple $/\mathbb{C}$, $\langle\cdot,\cdot\rangle$ inv. sym. bilin form.
$\{a_i\}$ an ON basis of $\mathfrak{g}$.

$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} a_i \otimes a_i \in S^2\mathfrak{g}$ the Casimir.

from category $\mathcal{O}$.

$V_1, \ldots, V_n$ repns of $\mathfrak{g}$, $1 \leq i \neq j \leq n$,

$$\Omega_{ij} : V_1 \otimes \ldots \otimes V_n \longrightarrow V_1 \otimes \ldots \otimes V_n$$

given by Ω acting on V_i & on V_j.
Clearly $\Omega_{ij} = \Omega_{ji}$ then.

$[\Omega_{ij}, \Omega_{kl}] = 0$ if i,j,k,l distinct obviously.

$[\Omega_{ij} + \Omega_{ik}, \Omega_{jk}] = 0$ if i,j,k distinct easily.

$X_n \subset \mathbb{C}^n$ the config. space of n distinct pts in $\mathbb{C}$.
& $\mathcal{E}$ the trivial bundle on X_n with fibre $V_1 \otimes \ldots \otimes V_n$.
For $\hbar \in \mathbb{C}$, KZ connection on $\mathcal{E}$ is

$$\nabla_{KZ} = d - \underbrace{\hbar \sum_{i<j} \Omega_{ij} \frac{d(z_i - z_j)}{z_i - z_j}}_{\in \Omega^1(X_n, \mathrm{End}\,\mathcal{E})}.$$

Thm $\nabla_{KZ} = d - \hbar\omega_{KZ}$ is a flat connection

Pf $\mathrm{Curv}(\nabla_{KZ}) = \underbrace{-\hbar d\omega_{KZ}}_{\substack{\shortparallel \\ 0 \text{ b/c} \\ \omega_{KZ} \text{ is exact.}}} + \hbar^2 \underbrace{[\omega_{KZ}, \omega_{KZ}]}_{=0}$

$=0$ b/c of relations of Ω_{ij} & the identity $\frac{1}{(a-b)(b-c)} + \frac{1}{(b-c)(c-a)} + \frac{1}{(c-a)(a-b)} =$

What are flat sections? Need finite dim'l. $V_i \in \mathcal{O}$ is enough.

$\mathcal{E} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{E}_\lambda$, $\mathcal{E}_\lambda$ finite dim'l.

The equations for a flat section are the KZ equations.

$F(z_1, \ldots z_n) \in (V_1 \otimes \ldots \otimes V_n)[\beta] \quad (\beta \in \mathfrak{h}^*)$

$$\boxed{\frac{\partial F}{\partial z_i} = \hbar \sum_{j \neq i} \frac{\Omega_{ij} F}{z_i - z_j}} \leftarrow \boxed{\text{KZ equations}}$$

What are the solutions? (They exist on $\tilde{X}_n$)
$\pi_1(X_n, x) = PB_n$ (the pure braid group).

We get representations $PB_n \to \mathrm{Aut}(V_1 \otimes \ldots \otimes V_n)$. in this way.

Eg// $n=3$, $\mathfrak{g} = sl_2$, $V_i = M_{\mu_i}$ $(i=1,2,3)$

$\beta = \mu_1 + \mu_2 + \mu_3 - 2$.

There's a 2D space of h.wt. functions.
Solutions will be in terms of Gauss hypergeometric functions.