Now we'd like to write the Weyl-Kac formula more explicitly in the case of affine Lie algebras, relating it to theta-functions, and use it to construct the discrete series for the Virasoro algebra, finishing our study of the determinant formula for the Virasoro algebra and its unitary representations

First we need to discuss the Weyl group of affine Lie algebras. We'll see that for affine algebras, Weyl group action is a part of the action of the corresponding group.

Let $\mathfrak{g}$ be a simple Lie algebra. $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$. We have an extension $\tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$. It's a bit smaller than $\mathfrak{g}_{ext}$ (A affine) but equivalent for our purposes. The algebra $\tilde{\mathfrak{g}}$ carries an invariant nondegenerate form:

On $L\mathfrak{g}$:

(a, b) = constant term $(a(t), b(t))$ (where $(,)$ is a form on $\mathfrak{g}$ with $(\alpha, \alpha) = 2$ for long roots)

We extend it to $\tilde{\mathfrak{g}}$ by $(K, d) = 1$, $(K, K) = (d, d) = (K, L\mathfrak{g}) = (d, L\mathfrak{g}) = 0$.

Prop. This form is invariant (unique up to scale with $(d,d)=0$)

Pf. Easy exercise.

Now let G be the simply connected complex Lie group corresponding to $\mathfrak{g}$ (e.g. for $\mathfrak{g}=\mathfrak{sl}_n$, $G=SL_n$). G is defined by algebraic equations in $Mat_N(\mathbb{C})$, and $LG = G[t,t^{-1}]$ is defined by the same ~~equations~~ in $Mat_N(\mathbb{C})[t,t^{-1}]$.

The group LG acts on $L\mathfrak{g}$ by conjugation. Also it acts on $L\mathfrak{g}\oplus\mathbb{C}d$.

Prop. The action of LG on $L\mathfrak{g}$ uniquely extends to $L\mathfrak{g}\oplus\mathbb{C}d$ by

$$g(t)\circ d = d + g(t)\cdot t\frac{d}{dt}\left(g^{-1}(t)\right) = d + g\cdot(g^{-1})' = d - g'g^{-1}$$

(i.e. we think of d as a connection on $\mathbb{C}^*$).

Pf. We have $[d, a(t)] = ta'(t)$.

So $[g\circ d, g\circ a] = \left[d + gt\frac{d}{dt}(g^{-1}),\ gag^{-1}\right]$

$$= t\partial_t(gag^{-1}) + t\,g(g^{-1})'gag^{-1} - t\,ga(g^{-1})'$$

$$= \cancel{tg'ag^{-1}} + tga'g^{-1} + \cancel{tga(g^{-1})'} - \cancel{tg'ag^{-1}} - \cancel{tga(g^{-1})'}$$

$$= g\circ[d,a]$$

The action of LG on $\mathbb{C}d \oplus L\mathfrak{g}$ lifts to an action on $\tilde{\mathfrak{g}}$ by

$$g \circ K = K$$

$$g \circ X = gXg^{-1} + \operatorname{Res}_{t=0}(g'Xg^{-1})\,dt \cdot K$$

$$g \circ d = d - g'g^{-1} - \tfrac{1}{2}(g'g^{-1}, g'g^{-1}) \cdot K$$

Moreover, this action preserves the bilinear form. ("Lifts" means that on the quotient by K it gives the old action).

Proof. Explicit calculation e.g.

$$(g\circ d, g\circ d) = \Big(d - g'g^{-1} - \tfrac{1}{2}(g'g^{-1}, g'g^{-1})K\,,\ d - g'g^{-1} - \tfrac{1}{2}(g'g^{-1}, g'g^{-1})K\Big) = (g'g^{-1}, g'g^{-1}) - (g'g^{-1}, g'g^{-1}) = 0.$$

Now let us specialize to $\mathfrak{g} = \mathfrak{sl}_2$.

We have $\tilde{\mathfrak{h}} = \mathbb{C}d \oplus \mathbb{C}K \oplus \mathbb{C}\alpha$, where $(\alpha, \alpha) = 2$ (we identify $\tilde{\mathfrak{h}}$ with $\tilde{\mathfrak{h}}^*$ and α with h_α). We have: $(\alpha,\alpha) = 2$, $(K, d) = 1$, other products are zero

$$h_1 = \alpha,\quad h_0 = K - \alpha$$

$$\omega_0 = d,\quad \omega_1 = \tfrac{1}{2}\alpha + d \quad \text{so} \quad \omega_i(h_j) = \delta_{ij}.$$

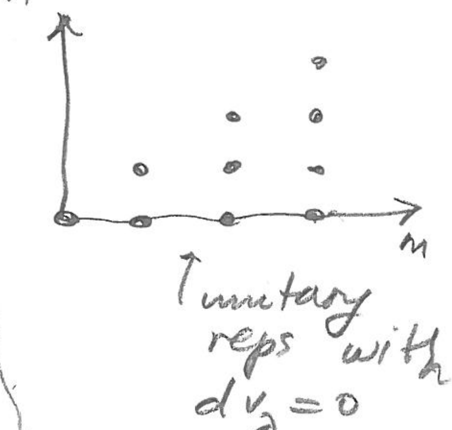
Weights $\lambda = md + \frac{n}{2}\alpha + zK = n\omega_1 + (m-n)\omega_0 + zK$

L_λ is integrable $\Leftrightarrow m, n \in \mathbb{Z}_+$ and $m \geq n$

L_λ is unitary $\Leftrightarrow \underset{m \geq n}{m, n \in \mathbb{Z}_+}$, and $z \in \mathbb{R}$

$K/L_\lambda = m$ (level of representations)

Now we want to define the Weyl group of $\tilde{\mathfrak{g}}$.



<u>Def.</u> $\widetilde{W} = \{ g \in LG \mid g \tilde{\mathfrak{h}} g^{-1} \subseteq \tilde{\mathfrak{h}} \}$

W = the image of $\widetilde{W}$ in $\mathrm{Aut}_{\mathbb{C}}(\tilde{\mathfrak{h}})$.

One can show it is the same as previous definition of the Weyl group that we gave.

Let us compute this for $\mathfrak{sl}_2$

$$g \circ \alpha = g \alpha g^{-1} + \mathrm{Res}(g' \alpha g^{-1}) \cdot K \in \tilde{\mathfrak{h}}$$

$$\Rightarrow g \alpha g^{-1} \in \mathfrak{h},$$

Also so $g \alpha g^{-1} = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}$, $\det = -\beta^2$, so $\beta = \pm 1$

So two cases:

1) $g \alpha g^{-1} = \alpha$, g diagonal $\left(g = \begin{pmatrix} a(t) & 0 \\ 0 & \frac{1}{a(t)} \end{pmatrix} \right)$

2) $g \alpha g^{-1} = -\alpha$, $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot$ diagonal.

But if $a, \frac{1}{a}$ are both Laurent polynomials then we have

(1) $t_m^+ = \begin{pmatrix} c t^m & 0 \\ 0 & c^{-1} t^{-m} \end{pmatrix}$ $\quad t^+ = t_m$ $\qquad$ (2) $t_m^- = \begin{pmatrix} c t^m & 0 \\ 0 & c t^{-m} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Note that these elements of W don't depend on c since $\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ acts trivially on $\tilde{\mathfrak{h}}$

$\sigma = Ad\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

We see that the Weyl group of $\tilde{\mathfrak{g}}$ is: $\langle t_m, \sigma t_m, m \in \mathbb{Z} \rangle$ with obvious multiplication rule.

We have

$$\sigma(\alpha) = -\alpha$$
$$\sigma(K) = K$$
$$\sigma(d) = d$$
$$t_m(\alpha) = \alpha + 2mK$$
$$t_m(K) = K$$
$$t_m(d) = d - m\alpha - m^2 K.$$

So now we can write the character formula more explicitly. Recall that the Kac character formula says that if λ is an integrable weight then

$$ch_{L_\lambda}(h) \overset{def}{=} Tr_{L_\lambda}(e^h) = \frac{\sum_{w \in W} \det(w) e^{(w(\lambda+\rho), h)}}{\sum_{w \in W} \det(w) e^{(w\rho, h)}}$$

Here $h = 2\pi i\left(\frac{1}{2} z\alpha - \tau d + uK\right)$

We have $\det(t_m) = 1$, $\det(t_m \sigma) = -1$

$$\rho = \omega_0 + \omega_1 = 2d + \frac{1}{2}\alpha$$

The formula gives a formal series in exponentials of z, τ, u.

But one can show that they converge to analytic functions in appropriate regions.

We introduce theta functions

$$\theta_{n,m}(\tau, z, u) = e^{2\pi i m u} \sum_{k \in \frac{n}{2m} + \mathbb{Z}} e^{2\pi i m(k^2\tau + kz)}$$

$n \in \mathbb{Z}, m \in \mathbb{Z}_+$

if we replace $x = e^{2\pi i z}$, $q = e^{2\pi i \tau}$ then the sum is

$$\sum x^{mk} q^{mk^2/2}$$

So the Kac formula says:

$$ch_\lambda(h) = q^{-s_\lambda} \frac{\theta_{n+1,m+2}(\tau, z, u) - \theta_{-n-1,m+2}(\tau, z, u)}{\theta_{1,2}(\tau, z, u) - \theta_{-1,2}(\tau, z, u)}$$

where $q = e^{2\pi i \tau}$ and $s_\lambda = \frac{(n+1)^2}{4(m+2)} - \frac{1}{8} + r$

<u>Proof.</u> The sums over t_m and $t_m\sigma$ actions convert to theta-functions.

<u>Example</u>. Consider the "basic representation", $m = 1, n = 0$, $\lambda = \omega_0 = d$

$$s_\lambda = \frac{(n+1)^2}{4(m+2)} - \frac{1}{8} + r = \frac{1}{12} - \frac{1}{8} = \frac{-1}{24} \quad (\text{if } r = 0).$$

So

$$ch_\lambda(h) = q^{-\frac{1}{24}} \frac{\Theta_{1,3}(\tau,z,u) - \Theta_{-1,3}(\tau,z,u)}{\Theta_{1,2}(\tau,z,u) - \Theta_{-1,2}(\tau,z,u)}$$

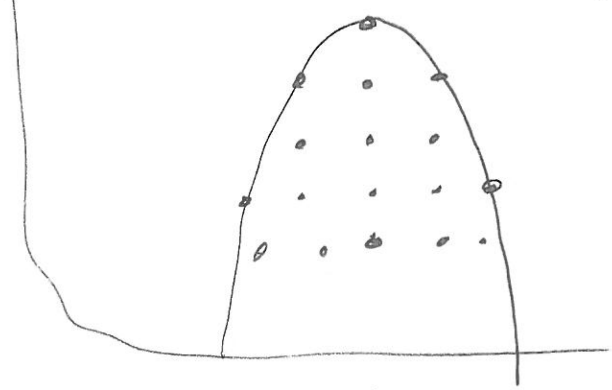
and using product formulas for theta functions we get

$$ch_\lambda(h) = \frac{\Theta_{0,1}(\tau,z,u)}{\prod_{n\geq 1}(1-q^n)}.$$

(this is also clear from the vertex operator construction from homework).

$(= ch\, \mathcal{B}_{even} = ch\, F_{even})$.

Now we want to construct the discrete series for Virasoro. Consider the tensor product



$L_d \otimes L_\lambda$.

It is of level $m+1$ and is unitary.

So $L_d \otimes L_\lambda = \bigoplus L_\mu$,

where μ are integrable weights of level $m+1$. We would like to find the multiplicities. For these purposes we can just multiply characters.

Prop. $$ch_d(h)\, ch_\lambda(h) = \sum_{\substack{\ell\in\mathbb{Z}\\ (\text{finite})}} \psi^{(q)}_{m,n,\ell}\, ch_{d+\lambda-\ell\alpha}$$

where

$$\psi_{m,n,\ell} = \frac{f_\ell^{m,n} - f_{n+1-\ell}^{m,n}}{\varphi(q)}$$

and

$$f_\ell^{m,n} = \sum_{j\in\mathbb{Z}} q^{(m+2)(m+3)j^2 + ((m+1+2\ell(m+2))j + \ell^2}$$

(explicit computation).

Corollary

$$\mathrm{Hom}_{\widehat{sl}_2}\big(L(d+\lambda-\ell\alpha),\, L(d)\otimes L(\lambda)\big) = U_{m,n,\ell},$$

where

$$\mathrm{tr}\big|_{U_{m,n,\ell}}(q^d) = \psi_{m,n,\ell}.$$

In particular, $\psi_{m,n,\ell}$ has positive coefficients.

Now, we know from the coset construction that $U_{m,n,\ell}$ is a Vir representation with $c = 1 - \frac{6}{(m+2)(m+3)}$; also we can compute the lowest eigenvalue of L_0 on this representation, and it is $h_{r,s}(m)$. So we see that $L(c_m, h_{r,s}(m))$ is unitary (actually $U_{m,n,\ell}$ is exactly the irreducible representation)

$r = n+1,\ s = n+1-2\ell,\ \ell \geq 0$

$r = m-n+1,\ s = m-n+2+2\ell,\ \ell < 0$

$c_m = 1 - \frac{6}{(m+2)(m+3)}$

Also we see that

$$\frac{q^{h_{r,s}(m)}}{\varphi(q)} - \mathrm{ch}\, U_{m,n,\ell} \sim \frac{q^{h_{r,s}(m)+rs}}{\varphi(q)}$$

$$\left(\text{i.e. } \mathrm{ch}\, U_{m,n,\ell} = \frac{q^{h_{r,s}(m)}}{\varphi(q)}\left(1 - q^{rs} + \ldots\right)\right)$$

which means that $M_{c_m, h_{r,s}(m)}$ has a singular vector of degree rs, and this finishes the proof of the Kac determinant formula.