Now we'd like to write the Weyl-Kac formula more explicitly in the case of affine Lie algebras, relating it to theta functions, and use it to construct the discrete series for the Virasoro algebra, finishing our study of the determinant formula for the Virasoro algebra and its unitary representations.

First we need to discuss the Weyl group of affine Lie algebras. We'll see that for affine algebras, Weyl group action is a part of the action of the correspondingly group.

Let $g$ be a simple Lie algebra $\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$. We have an extension $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{c}K \oplus \mathfrak{cd}$. It's a bit smaller than $\mathfrak{g}_{\text{ext}}[\mathfrak{aff}]$ but equivalent for our purposes. The algebra $\tilde{\mathfrak{g}}$ carries an invariant nondegenerate form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$

\[ (\alpha, \beta) = \text{constant term} \ (a(t), b(t)) \] (where $1$)

We extend it to $\tilde{\mathfrak{g}}$ by $(K,d) = 1$, $(K,K) = (d,d) = (K,\xi) = (d,\xi) = 0$.

$(\alpha, \alpha) = 2$ for long roots.
Prop. This form is invariant (uniquely with \( (d/d) = 0 \))

Pf. Easy exercise.

Now let \( G \) be the simply connected complex Lie group corresponding to \( \mathfrak{g} \) (e.g. for \( \mathfrak{g} = \mathfrak{sl}_n, \ G = \mathbb{S} \mathfrak{l}_n \)). \( G \) is defined by algebraic equations in \( \text{Mat}_N \mathbb{C} \), and \( \mathbb{C}[t, t^{-1}] \) is defined by the same equations in \( \text{Mat}_N \mathbb{C}[t, t^{-1}] \).

The group \( LG \) acts on \( \log \) by conjugation. Also it acts on \( \log \otimes \mathfrak{c} \mathfrak{d} \).

Prop. The action of \( LG \) on \( \log \) uniquely extends to \( \log \otimes \mathfrak{c} \mathfrak{d} \) by

\[
g(t) \circ d = d + g(t) \cdot \frac{dt}{dt} (g^{-1}(t)) = d + g(g^{-1})' = d - g' g^{-1}
\]

(i.e. we think of \( \mathfrak{d} \) as a connection on \( \mathfrak{c} \mathfrak{d} \)).

Pf. We have \( [d, a(t)] = ta'(t) \).

So

\[
[g \circ d, g \circ a] = [d + g(t) d(g^{-1}), g a g^{-1}]
\]

\[
= t \frac{d}{dt} (g a g^{-1}) + t g(g^{-1})' g a g^{-1} - t g a (g^{-1}'),
\]

\[
= tg' a g^{-1} + tg a' g^{-1} + t g a (g^{-1})' - t g' a g^{-1} - t g a (g^{-1})',
\]

\[
= g \circ [d, a]
\]
The action of $LG$ on $C \otimes \mathbf{g}$ lifts to an action on $\mathbf{g}$ by

$$g_0K = K$$

$$g_0X = gXg^{-1} + \int_{t=0}^\infty (g'Xg^{-1}) \, dt \cdot K$$

$$g_0d = d - g'g^{-1} - \frac{1}{2} (g'g^{-1}, g'g^{-1}) \cdot K$$

Moreover, this action preserves the bilinear form. ("Lifts" means that the quotient of $C \otimes \mathbf{g}$ by $K$ on the quotient gives the old action.)

Proof. Explicit calculation gives the proof.

$$\langle g_0d, g_0d \rangle = (d - g'g^{-1} - \frac{1}{2} (g'g^{-1}, g'g^{-1}) \cdot K, d - g'g^{-1} - \frac{1}{2} (g'g^{-1}, g'g^{-1}) \cdot K) = \langle g'g^{-1}, g'g^{-1} \rangle - \langle g'g^{-1}, g'g^{-1} \rangle = 0$$

Now let us specialize to $g = h_2$. We have $\mathfrak{g} = C \otimes \mathfrak{c} \oplus C \mathfrak{k} \oplus \mathfrak{c} \mathfrak{x}$, where $\langle x, x \rangle = 2, \langle k, d \rangle = 1, \langle h_0, h_0 \rangle = 0$.

Let $h_1 = d, h_0 = K - x$

$$w_0 = d, w_1 = \frac{1}{2} x + d$$

so

$$\langle w_i, (h_j) \rangle = \delta_{ij}$$

Weights $\lambda = md + \frac{n}{2} x + z k = nw_1 + (m-n)w_0 + zk$

$\lambda$ is integrable ($\Rightarrow$) $m, n \in \mathbb{Z}_+$ and $m \geq n$

and $\lambda$ is unitary ($\Rightarrow$) $m, n \in \mathbb{Z}_+$, and $r < R$
\[ K/L \cong m \text{ (level of representations)} \]

Now we want to define the Weyl group of \( \mathfrak g \).

**Def:** \( \tilde{W} = \{ g \in L G \mid g \tilde{h} g^{-1} \leq \tilde{\mathfrak h} \} \)

\( W \) = the image of \( \tilde{W} \) in \( \text{Aut}(\mathfrak h) \).

One can show it is the same as previous definition of the Weyl group that we gave.

Let us compute this for \( H_\mathbb{R} \):

\[ g \circ \alpha = g \circ \alpha g^{-1} + \text{Res} \left( g' \alpha g^{-1} \right) \cdot K \subset \tilde{\mathfrak h} \]

\( \Rightarrow g \alpha g^{-1} \in \tilde{\mathfrak h} \),

so \( g \alpha g^{-1} = (\beta \circ 0) \), \( \det = -\beta^2 \), so \( \beta = \pm 1 \)

So two cases:

1) \( g \alpha g^{-1} = \alpha \), \( g \) diagonal \( (g = \left( \begin{array}{cc} a(t) & 0 \\ 0 & 1 \end{array} \right) ) \)

2) \( g \alpha g^{-1} = -\alpha \), \( g = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \) diagonal.

But if \( a, -a \) are both Laurent polynomials then we have

\[ 1 \rightarrow t^m \rightarrow t_n \rightarrow 0 \rightarrow \rightarrow t_{-m} \rightarrow 0 \]

\[ (1) \quad t^m = (c t^m, 0) \quad (2) \quad t_{-m} = (0, c t^{-m}) \]

Note that these elements of \( W \) don't depend on \( c \) since \( (c, 0) \) acts trivially.
\( \sigma = \text{Ad}(\theta) \)

We see that the Weyl group of \( \bar{G} \)
is: \( \langle t_m, s t_m, m \in \mathbb{Z} \rangle \) with obvious multiplication rule.

We have

\[
\begin{align*}
\sigma(\alpha) &= -\alpha \\
\sigma(K) &= K \\
\sigma(d) &= d \\
t_m(\alpha) &= \alpha + 2mK \\
t_m(K) &= K \\
t_m(d) &= d - m\alpha - m^2K.
\end{align*}
\]

So now we can write the character formula more explicitly. Recall that the Kac character formula says that if \( \alpha \) is an integrable weight then

\[
\text{ch}_{L_{\lambda}}^G(h) = \frac{\text{Tr}_{L_{\lambda}}(e^h)}{\text{Tr}_{L_{\lambda}}(e^0)} = \sum_{w \in W} \frac{\text{det}(w) e^{w(\alpha + p), h}}{\text{det}(w) e^{w(p, h)}}
\]

Here \( h = 2\pi i \left( \frac{1}{2} \right) d - \pi d + \nu K \)

We have \( \text{det}(t_m) = 1 \), \( \text{det}(t_m \sigma) = -1 \)

\( p = \omega_0 + \omega_1 = 2d + \frac{1}{2} \alpha \)

The formula gives a formal series in exponentials of \( z, t, u \).
But one can show that they converge to analytic functions in appropriate regions.

We introduce theta functions
\[ \Theta_{n,m} (\tau, z, u) = e^{2\pi i m u} \sum_{n \in \mathbb{Z}, m \in \mathbb{Z}_+} e^{2\pi i m (k^2 \tau + k z)} \]

if we replace \( x = e^{2\pi i \tau} \), \( q = e^{2\pi i \tau} \), then the sum is
\[ \sum_{m, k} x^m q^{n k^2/2} \]

So the Kac formula says:
\[ \chi_n(h) = q^{-5/2} \frac{\Theta_{n+1, m+2} (\tau, z, u) - \Theta_{-n-1, m+2} (\tau, z, u)}{\Theta_{1, 2} (\tau, z, u) - \Theta_{-1, 2} (\tau, z, u)} \]

where \( q = e^{2\pi i \tau} \) and \( \frac{5}{2} = \frac{(n+1)^2}{4(m+2)} - \frac{1}{8} + \tau \)

Proof: The sums over \( t_m, t_{-m} \) become convergent to theta-functions.

Example. Consider the “basic representation”, \( m = 1, n = 0 \), \( \lambda = \omega_0 = 0 \)
\[ S_{\lambda} = \frac{(n+1)^2}{4(m+2)} - \frac{1}{8} + \tau = \frac{1}{12} - \frac{1}{8} = \frac{1}{24} \quad (\text{if } r=0) \]

So
\[ ch_\chi(h) = q^{-\frac{1}{24}} \frac{\Theta_{1,3}(t, z, u) - \Theta_{-1,3}(t, z, u)}{\Theta_{1,2}(t, z, u) - \Theta_{-1,2}(t, z, u)} \]

and using product formulas for theta functions we get

\[ ch_\chi(h) = \frac{\Theta_{0,1}(t, z, u)}{\prod_{n=1}^{\infty} (1 - q^n)} \]

\[ (= ch B_{\text{even}} = ch F_{\text{even}}) \]

Now we want to construct from the vertex operator construction the discrete series for \( V_{\text{mod}} \). Consider the tensor product

\[ L_d \otimes L_2 \]

It is of level \( m+1 \) and is unitary. So

\[ L_d \otimes L_2 = \bigoplus L_\mu \]

where \( \mu \) are integrable weights of level \( m+1 \). We would like to find the multiplicities. For these purposes, we can just multiply characters.
Prop. \( \text{ch}_d(h) \cdot \text{ch}_\lambda(h) = \sum_{\ell \in \mathbb{Z}} \psi_{m,n,\ell}^{(\phi)} \text{ch}_{d+\ell} - \ell \alpha \) (finite)

where

\[ \psi_{m,n,\ell} = \frac{f_{\ell} - f_{\ell+1} - \ell}{\varphi(q)} \]

and

\[ f_{\ell} = \sum_{j \in \mathbb{Z}} q^{(m-2)(m+3)} j^2 \left( \frac{-(m+2)\ell+2}{1} \right) + \ell^2 \]

(explicit computation).

Corollary.

\( \text{Hom}_{\mathfrak{g}_\mathbb{C}}(\mathcal{L}(d+\lambda - \ell \alpha), \mathcal{L}(d) \otimes \mathcal{L}(\lambda)) = U_{m,n,\ell} \)

where

\[ \text{tr} U_{m,n,\ell}(q^d) = \psi_{m,n,\ell} \]

In particular, \( \psi_{m,n,\ell} \) has positive coefficients.

Now, we know from the coset construction that \( U_{m,n,\ell} \) is a Vz representation with

\[ c = \frac{1}{(m+2)(m+3)} \]

also we can
compute the lowest eigenvalue of $L_0$ on this representation, and it is $h_{r,s}(m)$. So we see that $L((m, h_{r,s}(m))$ is unitary (actually $\psi_{mn}$ is exactly the irreducible representation). Also we see that

$$\frac{q^{h_{r,s}(m)}}{\varphi(q)} = ch_{mn,e} \sim \frac{q^{h_{r,s}(m)} + rs}{\varphi(q)}$$

(i.e. $ch_{mn,e} = \frac{q^{h_{r,s}(m)}}{\varphi(q)} (1 - q^{rs} + \ldots)$)

which means that $M_{c, h_{r,s}(m)}$ has a singular vector of degree $rs$, and this finishes the proof of the Kac determinant formula.