Lecture 2

Let us now pass to representation theory. We start with Dixmier's lemma, which is an infinite dimensional version of Schur's lemma.

Lemma. Let V be a countably dimensional irreducible module over an algebra A over $\mathbb{C}$. Then any morphism $\phi : V \to V$ is a scalar. In particular, this holds if A is countably dimensional.

Proof. Consider the algebra $D = \mathrm{End}_A V$. D is a division algebra, and it is countably dimensional since $\forall v \in V$, $v \neq 0$, v generates V and so $\phi \in D$ is completely determined by $\phi(v) \in V$ (so $D \subset V$). Now, if $\phi \notin \mathbb{C}$ then ϕ is transcendental$/\mathbb{C}$ so $\mathbb{C}(\phi) \subset D$, but $\mathbb{C}(\phi)$ is uncountably dimensional ($\frac{1}{\phi - \lambda}$, $\lambda \in \mathbb{C}$, are linearly independent).

Corollary. In the setting of the lemma, if $C \in A$ is a central element then $C|_V$ is a scalar.

Now let us discuss the representation theory of the Heisenberg algebra $\mathcal{A}$. We are interested in irreducible representations. In such a representation, K acts by a certain scalar k (by Dixmier's Lemma). If $k=0$, we get a representation of the abelian Lie algebra $\mathcal{A}_0 = \mathcal{A}/\mathbb{C}K$, which is 1-dimensional. So let us consider the more interesting case $k \neq 0$. It suffices to consider $k=1$, since $\mathcal{A}$ has an automorphism sending K to λK for any $\lambda \neq 0$ ($a_i \mapsto \lambda a_i$ for $i>0$, $a_i \mapsto a_i$ for $i \leq 0$). If K acts by 1 on some representation V, then V is a representation of

$$U(\mathcal{A})/(K-1).$$

<u>Proposition</u>. We have an isomorphism

$$\xi: U(\mathcal{A})/(K-1) \to \mathrm{Diff}(x_1, x_2, \dots) \otimes \mathbb{C}[x_0],$$ where

$\mathrm{Diff}(x_1, x_2, \dots)$ is the algebra of differential operators in variables $x_1, x_2, \dots$ with polynomial coefficients. It is given by the formula $\xi(a_{-j}) = x_j$, $\xi(a_j) = j\frac{\partial}{\partial x_j}$ for $j \geq 1$, $\xi(a_0) = x_0$, $\xi(K) = 1$.

<u>Proof</u>. It is clear that ξ is a well defined homomorphism, which is surjective, since all the generators are in the image. Also, by PBW theorem (easy part), $U(\mathcal{A})/(K-1)$ is spanned by elements

$\prod_{i \in \mathbb{Z}} a_i^{m_i}$, so ξ is injective, as the images of these elements are linearly independent.

<u>Corollary</u>. For every $\mu \in \mathbb{C}$, we have a module F_μ over $\mathcal{A}$, $F_\mu = \mathbb{C}[x_1, x_2, \dots]$, where differential operators act as usual, and x_0 acts by multiplication by μ

Def. The representation F_μ is called the Fock representation.

Proposition. The modules F_μ are irreducible and pairwise non-isomorphic.

Proof. It is clear that they are not isomorphic since x_0 acts by different eigenvalues. To prove irreducibility, note that F_μ is generated by 1, and for any $P \in F_\mu$, there exists a (monomial) differential operator D such that $DP = 1$. Namely, if P has monomial $\alpha x_1^{n_1} \cdots x_k^{n_k}$ of largest degree ($\alpha \neq 0$), then we can take $D = \frac{1}{\alpha} \frac{\partial_1^{n_1}}{n_1!} \cdots \frac{\partial_k^{n_k}}{n_k!}$, where $\partial_j = \frac{\partial}{\partial x_j}$ $\left(= \frac{1}{j} a_j\right)$ (D will kill all the other monomials in P).

It is not true that F_μ are the only irreducible modules over $\mathcal{A}$ with $K=1$, but they are the only irreducible modules in which $\mathbb{C}[a_1, a_2, \ldots]$ acts locally nilpotently. Namely, we have

Proposition (i) Let V be an irreducible module over $\mathcal{A}$ in which $K=1$, $a_0=\mu$, and such that for any $v\in V$, $\mathbb{C}[a_1,a_2,\ldots]v$ is finite dimensional, and a_i, $i>0$ act in this space by nilpotent operators. Then $V\cong F_\mu$.

(ii) Let V be any module as in (i) (not necessarily irreducible) such that $\forall v\ \exists N\ \forall i\geq N\ a_iv=0$. assume $a_0v=\mu v$, $v\in V$. Then V is $F_\mu\otimes U$, where U is a vector space.

Proof (i) Let $v\in V$, $v\neq 0$, and $W=\mathbb{C}[a_1,a_2,\ldots]v$. Then W has a vector $w\neq 0$ such that $a_iw=0$, $i>0$, and $a_0w=\mu w$. It is easy to see that we have a homomorphism $\gamma: F_\mu\to V$ such that $\gamma(1)=w$. It is nonzero, so must be an isomorphism.

(ii) Let $v\in V$, $v\neq 0$, and $I_v\subset\mathbb{C}[a_1,a_2,\ldots]$ be the annihilator of v. Then $W=\langle v\rangle_{\subset V}$ is a quotient of $D(x_1,x_2,\ldots)/D(x_1,x_2,\ldots)I_v$, $= D(x_1,x_2,\ldots)\otimes_{\mathbb{C}[x_1,x_2,\ldots]}\mathbb{C}[x_1,\ldots]/I_v$. so it is an extension of finitely many copies of F_μ (length $=\dim\mathbb{C}[a_1,\ldots]/I_v$). Thus the

the Euler operator $E=\sum i x_i \frac{\partial}{\partial x_i}$ $= \sum_{i\geq 1} u_{-i} a_i$ acts on V locally finitely. Clearly, if $a_i v = 0$ $\forall i \geqslant 1$ then $Ev = 0$.

Conversely, if $E^m v = 0$ then if $a_i v \neq 0$, then $[E+i]^m a_i v = 0$ (as $[E, a_i] = -i a_i$) which contradicts (eigenvalues of E are ≥ 0). local nilpotence. So we see that $V_0 = \{v \mid a_i = 0\}$ coincides with the generalized 0-eigenspace of E, i.e., this generalized eigenspace is the usual eigenspace. This shows that the natural map $F_\mu \otimes V_0 \to V$ is an isomorphism. (the quotient must live in strictly positive degrees with respect to E). ∎

The module F_μ should be considered as a graded module. Namely, we have a $\mathbb{Z}$-grading $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$, $\mathcal{A}_i = \mathbb{C} a_i$ for $i \neq 0$ and $\mathcal{A}_0 = \langle a_0, K\rangle$, and $[\mathcal{A}_i, \mathcal{A}_j] \subseteq \mathcal{A}_{i+j}$. The module F_μ is graded in the sense that

$F_\mu = \bigoplus_{n\geqslant 0} F_\mu[n]$, where $F_\mu[n]$ is the space of polynomials of degree n $\left(\begin{smallmatrix}\text{vectors}\\ \text{of degree } -n\end{smallmatrix}\right)$ (where $\deg x_i = i$), and $A_i \otimes F_\mu[-n] \to F_\mu[-n+i]$.

$\dim F_\mu[-n] = p(n)$, the number of partitions of n, so the generating function

$$\sum \dim F_\mu[-n]\, q^n \overset{\text{``}Tr_{F_\mu}(q^E)\text{''}}{=} \frac{1}{(1-q)(1-q^2)\cdots}$$

It is usual to shift the grading by $\frac{\mu^2}{2}$, i.e. $\underset{\hat{F}_\mu}{\deg(\underset{\cap}{1})} = \frac{-\mu^2}{2}$ (for reasons to be explained later). Then

$$ch\, F_\mu = \sum \dim F_\mu[-n]\, q^{n+\frac{\mu^2}{2}} = \frac{q^{\mu^2/2}}{\prod_{i\geq 1}(1-q^i)}.$$

This is actually a special case of the more general representation theory of $\mathbb{Z}$-graded Lie algebras that we will now develop.

<u>Def.</u> A $\mathbb{Z}$-graded Lie algebra is a Lie algebra $\mathfrak{g}$ with a decomposition $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ and $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$.

We will deal with a special kind of such algebras.

<u>Def.</u> A $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ is said to be nondegenerate if 1) $\mathfrak{g}_n$ are finite dimensional for all n

2) $\mathfrak{g}_0$ is abelian

3) for $n \in \mathbb{N}$ and a generic $\lambda \in \mathfrak{g}_0^*$, the pairing $\mathfrak{g}_n \times \mathfrak{g}_{-n} \to \mathbb{C}$ given by $(x, y) \longmapsto \lambda([xy])$ is nondegenerat (in particular, $\dim \mathfrak{g}_n = \dim \mathfrak{g}_{-n}$)

<u>Examples.</u> It is easy to see that $\mathcal{A}$ and W, Vir are nondegenerate. Also a simple Lie algebra $\mathfrak{g}$ is nondegenerate with principal grading $\deg(e_i) = 1$, $\deg(f_i) = -1$, $\deg \mathfrak{h} = 0$. Finally, $\mathfrak{g}[t, t^{-1}]$ and affine Lie algebra $\hat{\mathfrak{g}}$ is nondegen-

nerate if we take the following grading: $\deg(e_i)=1$, $\deg(f_i)=-1$, $\deg(\mathfrak{h})=0$, $\deg(tf_\theta)=1$, $\deg(t^{-1}e_\theta)=-1$, $\theta=$ max root (exercise) (but $\mathfrak{g}[t,t^{-1}]$ without extension is not nondegenerate).

The reason we introduce this notion is because representation theory of such algebras looks especially nice.

Definition. The triangular decomposition of $\mathfrak{g}$ is $\mathfrak{g}=\mathfrak{g}_{<0}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{>0}$. Usually one denotes these subalgebras by $\mathfrak{n}_-$, $\mathfrak{h}$, $\mathfrak{n}_+$, so $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$.

Definition. Let $\lambda\in\mathfrak{h}^*$. The Verma module $M_\lambda^\pm$ over $\mathfrak{g}$ is the module $U(\mathfrak{g})\otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_\pm)}\mathbb{C}_\lambda$, where $\mathfrak{n}_\pm$ acts by 0 in $\mathbb{C}_\lambda$ and $\mathfrak{h}$ acts by λ. The module M_λ^+ will also be denoted by M_λ.

As a vector space, we have
$M_\lambda^+=U(\mathfrak{n}_-)v_\lambda^+$, $M_\lambda^-=U(\mathfrak{n}_+)v_\lambda^-$

where v_λ^+ is the highest weight vector satisfying $x v_\lambda^+ = \lambda(x) v_\lambda^+$, $x \in \mathfrak{h}$. This follows from the PBW theorem. Thus, we have a decomposition

$$M_\lambda^+ = \bigoplus_{n \geq 0} M_\lambda^+[-n],$$

where $M_\lambda^+[-n] = U(\mathfrak{n}_-)[-n] v_\lambda^+$, where $\mathcal{U}(\mathfrak{n}_-)[-n]$ is the subspace of degree $-n$. Moreover, we have

$$\sum \dim M_\lambda^+[-n] q^n = \frac{1}{\prod_{i > 0} (1-q^i)^{\dim \mathfrak{g}_{-i}}},$$

and similarly for M_λ^+.

Also, we have the following proposition.

<u>Prop</u>. There exists a unique $\mathfrak{g}$-invariant pairing $M_\lambda^+ \otimes M_{-\lambda}^- \to \mathbb{C}$ up to scaling. It restriction to $M_\lambda^+[-n] \otimes M_\lambda^-[m]$ is zero unless $n = m$.

Proof. $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+ \otimes M_{-\lambda}^-, \mathbb{C})$

$= \mathrm{Hom}_{\mathfrak{g}_{\geq 0}}(\mathbb{C}_\lambda \otimes M_{-\lambda}^-, \mathbb{C}) = \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}, \mathbb{C})$

$= \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$. So the form is unique up to scaling. If we normalize it by $(v_\lambda^+, v_{-\lambda}^-) = 1$, we get

$$(a v_\lambda^+, b v_{-\lambda}^-) = (v_\lambda^+, S(a) b v_{-\lambda}^-)$$

$$\overset{\parallel}{(S(b) a v_\lambda^+, v_{-\lambda}^-)}$$

where $a \in U(\mathfrak{n}_-)$, $b \in U(\mathfrak{n}_+)$ and S is the antipode. So for this to be nonzero, we need $\deg(a) \geq \deg(b)$ and $\deg(a) \leq \deg(b)$ if a, b are homogeneous. $\blacksquare$

Theorem. For any n, the form $(\ ,\ )|_{M_\lambda^+[-n] \otimes M_\lambda^-[n]}$ is nondegenerate for Zariski generic λ.