Recall $P = \mathfrak{h}^* \oplus F = \mathfrak{h}^*_{ext}$

We introduce the inner product $(\ ,\ )$ on P:

$$(\ ,\ ) : P \times P \longrightarrow \mathbb{C}$$

such that

$$(\underset{\mathfrak{h}^*}{\varphi} + \underset{P}{\alpha},\ \underset{\mathfrak{h}^*}{\psi} + \underset{P}{\beta}) = \varphi(h_\beta) + \psi(h_\alpha) + (h_\alpha, h_\beta)$$

This is nondegenerate since if we choose dual bases, the matrix is

$$\begin{pmatrix} 0 & I \\ I & D^{-1}A \end{pmatrix}.$$

(bases α_i, $h^*_{\alpha_i}$)

The inverse form on $\mathfrak{h}_{ext}$ has the form

$$(h_{\alpha_i}, h_{\alpha_j}) = d_i^{-1} a_{ij} = d_j^{-1} a_{ji}$$

$$(D_i, D_j) = 0, \quad (D_i, h_{\alpha_j}) = d_j\, \delta_{ij}$$

<u>Prop.</u> Let B be the extension of $(\ ,\ )$ on $\mathfrak{g}(A)$ to $\mathfrak{g}_{ext}(A)$. Then B is a nondegenerate form.

<u>Proof</u> Clear

Now we want to define the Casimir element

We define $\rho \in \mathfrak{h}^*$ by $\rho(h_i) = \frac{a_{ii}}{2}$ ($=1$ for Kac-Moody algebras)

So $(\rho,\alpha_i)=\rho(h_{\alpha_i})=d_i^{-1}\rho(h_i)=d_i^{-1}a_{ii}/2=\frac{1}{2}(\alpha_i,\alpha_i)$

$\Rightarrow (\rho,\rho)=0$ (since $\rho\in\mathfrak{h}^*$).

Recall that for simple Lie algebras

$$C=\sum_{\substack{a\in B\\ \uparrow\\ \text{orth. basis}}} a^2=\sum_{\substack{i\\ \uparrow\\ \text{orth. basis of Cartan}}} x_i^2+2h_\rho+2\sum_{\alpha>0} f_\alpha e_\alpha$$

Let us generalize it to any $\mathfrak{g}(A)$.

Let $\forall\alpha$, $e_\alpha^{(i)}$ be a basis of $\mathfrak{g}_\alpha$, $f_\alpha^{(i)}$ be the dual basis of $\mathfrak{g}_{-\alpha}$, so $(e_\alpha^{(i)},f_\alpha^{(j)})=\delta_{ij}$

We define

$$\Delta_+=2\sum_{\alpha>0}\sum_i f_\alpha^{(i)}e_\alpha^{(i)} : V\to V \quad \forall V\in\mathcal{O}$$

The sum is infinite, but becomes finite if we apply to any vector.

Also let $\Delta_0: V\to V$ $\Delta_0\big|_{V[\mu]}=(\mu,\mu+2\rho)$.

i.e. $\Delta_0=\sum_i x_i^2+2h_\rho$.

<u>Theorem</u> 1) $\Delta:=\Delta_++\Delta_0$ commutes with $\mathfrak{g}(A)$

2) Δ acts on $M(\lambda)$ by $(\lambda,\lambda+2\rho)$.

<u>Pf.</u> (2) follows from (1) easily, as

$\Delta v_\lambda=(\lambda,\lambda+2\rho)v_\lambda$, so $\Delta x v_\lambda=x\Delta v_\lambda=(\lambda,\lambda+2\rho)xv_\lambda$

Now we prove (i).

Clearly Δ commutes with h_i, so it suffices to check it commutes with e_i, f_i

We have $[\Delta_0, e_k] = +2h_{\alpha_k} e_k$, so we need to show that

$$[\Delta_+, e_k] = -2h_{\alpha_k} e_k.$$

We have

$$[\Delta_+, e_k] = 2\sum_{i,\alpha} [f_\alpha^{(i)} e_\alpha^{(i)} e_k - e_k f_\alpha^{(i)} e_\alpha^{(i)}]$$

$$= 2\sum_{i,\alpha} \left(f_\alpha^{(i)} [e_\alpha^{(i)}, e_k] - [e_k, f_\alpha^{(i)}] e_\alpha^{(i)} \right)$$

In the second sum, we have a term for $\alpha = \alpha_k$ which is exactly what we need:

$$[e_k, f_k] e_k d_k^{-1} = h_k e_k d_k^{-1} = h_{\alpha_k} e_k.$$

So we need to show that

$$\sum_{i,\alpha} f_\alpha^{(i)} [e_\alpha^{(i)}, e_k] - \sum_{i,\alpha \neq \alpha_k} [e_k, f_\alpha^{(i)}] e_\alpha^{(i)} = 0.$$

It suffices to show that $\forall \alpha$ such that $\alpha + \alpha_k$ is a pos. root,

$$\sum_i f_\alpha^{(i)} \otimes [e_\alpha^{(i)}, e_k] = \sum_i [e_k, f_{\alpha+\alpha_k}^{(s)}] \otimes e_{\alpha+\alpha_k}^{(s)}$$

Let's take the inner product with $e_\alpha^{(j)}$ in the first component. We get

$$[e_\alpha^{(j)}, e_K] \overset{?}{=} \sum \left(e_\alpha^{(j)}, [e_K, f^{(s)}_{\alpha+\alpha_K}]\right) e_\alpha^{(s)}$$

$$= \sum_\alpha \left([e_\alpha^{(i)}, e_K], f^{(s)}_{\alpha+\alpha_K}\right) e^{(s)}_{\alpha+\alpha_K} = [e_\alpha^{(j)}, e_K]$$

So the equality holds and we are done.

Remark: (note that in the affine case $\Delta = (K+h^\vee)L_0 - d$) (exercise). we recover Sugawara construction

Now we are in a position to develop representation theory.

Recall:

Prop. In a KM algebra $\mathfrak{g}(A)$, the Serre relations are satisfied.

Pf. It was proved before.

Def. Let $\mathfrak{g}$ be a Lie algebra, and V a $\mathfrak{g}$-module. Say that V is locally finite dim if $\forall v \in V$, $\dim U(\mathfrak{g})v < \infty$.

Say $v \in V$ is of finite type if $U(\mathfrak{g})v$ is f.d.

Exercise V is locally finite iff V is a sum of f.d. $\mathfrak{g}$-modules.

Def. Let $\mathfrak{g} = \mathfrak{g}(A)$ is a Kac-Moody algebra. We say that a $\mathfrak{g}$-module V is integrable if it is locally finite with respect to each subalgebra $\mathfrak{sl}(2)_i$.

(It is called integrable because it is integrable to the groups $(SL_2)_i$.)

Prop. $\mathfrak{g}$ is an integrable $\mathfrak{g}$-module.

Pf. lemma. Let $\mathfrak{a}$ be a Lie alg. with a Lie bracket invariant under another Lie algebra $\mathfrak{g}$ acting on $\mathfrak{a}$. If $a, b \in \mathfrak{a}$ are of finite type, then so is the commutator.

Pf. $U(\mathfrak{g})[a, b]$ is contained in $[U(\mathfrak{g})a, U(\mathfrak{g})b]$.

Pf of prop: ~~Enough~~ By lemma it is to check that generators of $\mathfrak{g}$ are of finite type. But this is true because of Serre rel. $U(\mathfrak{sl}_2)_i e_j$ is f.d. of dim $-a_{ij}$. The same for $U(\mathfrak{sl}_2)_i f_j$

<u>Prop.</u> If V is a $\mathfrak{g} = \mathfrak{g}(A)$ -module then V is integrable iff it has a set of generators v_j, $j=1,\dots,m$ which are of finite ~~type~~ under $(\mathfrak{sl}_2)_i$.

<u>Proof</u>. ~~Let~~ Fix i. Let $W_i \subset V$ be a subsp of $(\mathfrak{sl}_2)_i$ (f.d) containing all v_j. Then V, as an $(\mathfrak{sl}_2)_i$ -module, is a quotient of $\mathcal{U}(\mathfrak{g}) \otimes W_i$. But this module is locally finite under $(\mathfrak{sl}_2)_i$ since so is $\mathfrak{g}$.

<u>Prop.</u> Let $L(\lambda)$ be a highest weight irreducible module. Then:
$L(\lambda)$ is locally finite if and only if $\lambda(h_i) \in \mathbb{Z}_+ \ \forall i$.
(so for f.d. case, locally finite is the same as finite dimensional)

<u>Pf.</u> 1) $L(\lambda)$ locally finite $\Rightarrow v_\lambda$ of finite type for $(\mathfrak{sl}_2)_i$, and $e_i \cdot v_\lambda = 0$, $h_i \cdot v_\lambda = \lambda(h_i) v_\lambda \Rightarrow \lambda(h_i) \in \mathbb{Z}_+$.
2) If $\lambda(h_i) \in \mathbb{Z}_+$, it is easy to check that $e_j f_i^{\lambda(h_i)+1} v_\lambda = 0 \quad \forall j$ (including $j=i$).

so by irreducibility $f_i^{\lambda(h_i)+1} v_\lambda = 0$ and hence $U(sl_2)_i v_\lambda \cong \mathbb{C}^{\lambda(h_i)+1}$. So by the previous proposition, L_λ is integrable.

<u>Def</u>. The weights λ such that $\lambda(h_i) \in \mathbb{Z}_+$ are called dominant integrab weights.

Now we want to find the character of L_λ for dominant integrab λ.

For this we will need to discuss the Weyl group of the Kac-Moody algebra.

Define $r_i : P \to P$, $s_i(x) = x - x(h_i)\alpha_i$

<u>Prop</u>. (1) $(\,,\,)$ on P is invariant under r_i: $(r_i x, r_i y) = (x, y)$.

(2) If V is an integrable module then $\exists\ : V[\lambda] \to V[r_i\lambda]$ which is an isomorphism

(3) $r_i^2 = 1$.

<u>Proof</u>. (1) and (3) is easy. (3): $x - x(h_i)\alpha_i - x(h_i)\alpha_i + x(h_i)\alpha_i(h_i)\alpha_i = x$.

(2): We want to find isomorphisms $V[\lambda] \underset{E}{\overset{F}{\rightleftarrows}} V[r_i\lambda]$.

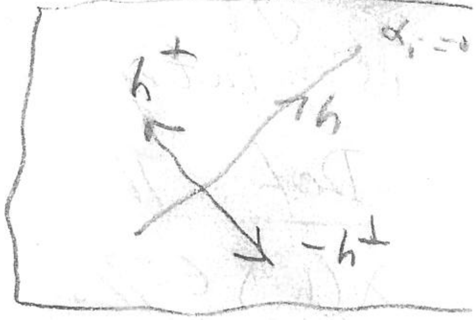
Let $\lambda(h_i) = m$. Then $m \in \mathbb{Z}$, and WLOG can assume that $m \geq 0$ (otherwise replace λ with $r_i\lambda$). Then can take $F = f_i^m$, $E = e_i^m$.

<u>Def</u>. The Weyl group $W \subset \mathrm{Aut}(P)$

is generated by r_i.

The elements r_i are called simple reflections. They are orthogonal reflections with respect to the hyperplane $\alpha_i=0$.

$r_i(h,h^\perp)=(h,-h^\perp)$



Rem. In general, W is not a finite group. It is finite only for semisimple Lie algebras.

Prop. $(\,,\,)$ is invariant under W.

Pf. Obvious since it is invariant with respect to the generators.

Prop. 1) If V is an integrable module then
$V[\mu]\cong V[w\mu]$ $\forall w\in W$ (isomorphic as vector spaces)

2) The set $R\subset Q$ of roots is W-invariant

3) $r_i(\alpha_i)=-\alpha_i$, and r_i defines a permutation of positive roots

Pf. 1) Follows from the case $w=s_i$

2) $R\cup\{0\}$ is the set of weights of the adjoint repr, which is integrable.

3) It is clear that $r(\alpha_i)=-\alpha_i$.

Now, $r(\alpha)=\alpha-\alpha(h_i)\alpha_i$, so it has a positive coeff. unless α is a multiple of α_i (and hence is a positive root). But the only multiple of α_i that is a positive root is α_i itself.

Now we are ready to discuss the character formula.

$\ldots (+ \ldots \lambda(h_i) \in \mathbb{Z}_+)$.

Theorem. Let χ be a dominant integral weight $(\chi(h_i) \in \mathbb{Z}_+)$, so $\chi \in P_+$. Let V be a h.w. module with h.w. χ, which is integrable. Then

1) V is isomorphic to the irreducible highest wt module with highest wt χ

2) $$\operatorname{ch} V = \sum_{w \in W} \det(w) \operatorname{ch} M(w(\chi+\rho)-\rho)$$

(Kac character formula).

So $$\operatorname{ch} V = \sum_{w \in W} \frac{\det(w)\, e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})^{\dim g_\alpha}}$$ (By the formula for character of Verma module)

Corollary. (Kac denominator formula)

$$\sum_{w \in W} \det(w)\, e^{w\rho-\rho} = \prod_{\alpha>0} (1-e^{-\alpha})^{\dim g_\alpha}$$

Proof. Set $\chi = 0$.

Proof of the theorem.

Lemma 1. 1) Let $\chi \in P_+$. Then $W\chi \subset D(\chi) = \{\chi - \sum k_i \alpha_i \mid k_i \geq 0\}$.

2) If $D \subset D(\chi)$ is a W-invariant set, then $D \cap P_+ \neq \emptyset$.

Proof. 1) Consider $L(\chi)$. Since $L(\chi)$ is locally finite, $P(L(\chi)) \overset{\text{def}}{=} \{\text{weights of } L(\chi)\}$ is W-inv.

But $\chi \in P(L(\chi))$, so $w\chi \subset P(L(\chi))$. So the statement follows since $P(L(\chi)) \subset D(\chi)$. (since χ is a highest wt).

2) Let $\psi \in D$, choose $w \in W$ s.t.
$$\chi - w\psi = \sum k_i \alpha_i \text{ has smallest possible}$$
$\sum k_i$. (Of course, $w\psi \in D(\chi)$)

We claim that $w\psi \in P_+$. Indeed, assume that $w\psi \notin P_+$. So $\exists\, i$ such that $(w\psi, \alpha_i) < 0$. But then let us take
$$r_i w\psi = w\psi - \underbrace{w\psi(h_i)}_{<0}\alpha_i .$$
$$\chi - r_i w\psi = \sum k_j \alpha_j + w\psi(h_i)\alpha_i \quad \text{and}$$
$$\sum k_j + w\psi(h_i) < \sum k_j . \Rightarrow\Leftarrow \text{ with minimality of } \sum k_i$$

Corollary 2 If $w \in W$, $w \neq 1$ then there is i such that $w\alpha_i < 0$

Proof. Choose $\chi \in P_+$ such that $w\chi \neq \chi$.

Then by lemma 1 (1)
$$w^{-1}\chi = \chi - \sum_i k_i \alpha_i ,$$
so again by Lemma 1 (1)
$$\chi = ww^{-1}\chi = w\chi - \sum_i k_i w\alpha_i = \chi - \sum k_i' \alpha_i - \sum_i k_i w\alpha_i ,$$
and $\sum k_i' > 0$. We have $\sum k_i' \alpha_i + \sum k_i w\alpha_i = 0$, So for some i, $w\alpha_i < 0$.

Prop. 3 If $\varphi, \psi \in P$
$\varphi(h_i) > 0 \quad \psi(h_i) \geq 0 \quad \forall i$
then $\quad w\varphi = \psi \iff \varphi = \psi, w = Id$

Proof $\varphi(h_i) > 0 \iff (\varphi, \alpha_i) > 0 \quad \forall i$
Suppose $\exists w \neq 1$ s.t. $w\varphi = \psi$. Then
By previous corollary 2 $\exists i$
s.t. $w\alpha_i < 0$. So

$$0 < (\varphi, \alpha_i) = (w^{-1}\psi, \alpha_i) = (\psi, w\alpha_i) \leq 0 \implies \Leftarrow$$

Prop. 4 Consider the Weyl–Kac denominator

$$K = e^{\rho} \prod_{\alpha > 0} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$$

Extend the action of W on the ring R where the characters live, by $w(e^{\lambda}) = e^{w(\lambda)}$

Then $\quad w(K) = \det(w) \cdot K.$

Proof. It suffices to prove for r_i.
But we have

$$r_i K = e^{r_i \rho} (1 - e^{\alpha_i}) \prod_{\substack{\alpha > 0 \\ \alpha \neq \alpha_i}} (1 - e^{-w(\alpha)})^{\dim \mathfrak{g}_\alpha}$$

(as r_i permutes all ~~p~~ roots other than α_i)

So $\dfrac{r_i K}{K} = e^{r_i \rho - \rho} \dfrac{1 - e^{\alpha_i}}{1 - e^{-\alpha_i}} = -1.$ (as $\rho(h_i) = 1$).
so $r_i \rho - \rho + \alpha_i = 0$

<u>Lemma 5</u> Let $\lambda, \nu \in P_+$ and suppose $\mu \in D(\nu)$, $\mu \neq \nu$. Then
$$(\nu+\rho)^2 - (\mu+\rho)^2 > 0.$$

<u>Proof</u>. $(\nu+\rho)^2 - (\mu+\rho)^2 = (\nu-\mu, \nu+\mu+2\rho)$

$\nu - \mu = \sum k_i \alpha_i$, so we get >0 since

$(\nu, \alpha_i) \geq 0$, $(\mu, \alpha_i) \geq 0$, $(\rho, \alpha_i) > 0$.

Now we can <u>prove the main theorem</u>. We prove part (2). Part (1) follows easily (since (2) applies to any highest weight module).

Recall that characters of Verma modules form a topological basis of the ring R. This means that there are constants $c_\lambda \in \mathbb{Z}$, $\lambda \in \bigcup_{i=1}^{n} D(\mu_i)$, such that
$$\operatorname{ch} V = \sum_\lambda c_\lambda \operatorname{ch} M_\lambda.$$
We saw this formally, now let us see this explicitly. Let $P(V) \subset \bigcup_{i=1}^{n} D(\mu_i)$. Let $S = \{i \mid \mu_i \in P(V)\}$. We have a homomorphism $\varphi : \bigoplus_{i \in S} d_i M_{\mu_i} \longrightarrow V$ which maps v_{μ_i} to some vector of weight μ_i (here $d_i = \dim V[\mu_i]$

let K be $\operatorname{Ker}\varphi$, and C the cokernel of φ. Then we have an exact sequence

$$0 \to K \to \bigoplus_{i \in S} d_i M_{\mu_i} \to V \to C \to 0.$$

So we have

$$\operatorname{ch} V = \sum_{i \in S} d_i \operatorname{ch} M_{\mu_i} - \operatorname{ch} K + \operatorname{ch} C,$$

and weights λ of K and C have the property $\mu_i - \lambda = \sum k_{ij}\alpha_j$, $\sum_j k_{ij} \geq 1$ for all. Now we can take all weights $\mu_i - \alpha_j$ which are in $P(C)$ and $P(K)$, and repeat the procedure. In the limit we get an expansion as above.

<u>Now we prove the theorem.</u>

We have a decomposition obtained as above:

$$\operatorname{ch} V = \sum_{\psi \in D(\chi)} c_\psi \operatorname{ch} M_\psi,$$

and $c_\chi = 1$ (as for $\psi \in D(\chi)$, $\psi \neq \chi$, $M_\psi[\chi] = 0$)

<u>Lemma 5</u> If $c_\psi \neq 0$ then $(\psi + \rho)^2 = (\chi + \rho)^2$.

<u>Proof.</u> Consider the Casimir Δ.

We know $\Delta|_V = (\chi, \chi+2\rho) = (\chi+\rho)^2 - \rho^2$.

<u>Claim 7</u> Any module W that appears in the proof of the decomposition has the same eigenvalue of Δ.

<u>Pf of Claim 7</u>. Clearly, if V has a certain eigenvalue γ of Δ, the so do $M(\mu_i)$, and so K and C have the same eigenvalue of Δ, so it is inherited in the procedure.

Now we proceed with proving the theorem.

<u>Lemma</u> 8 If $\psi+\rho = w(\chi+\rho)$ then

$$c_\psi = \det w \cdot c_\chi$$

<u>Pf</u>. We proved in Prop 4 that $wK = \det(w) K$, and also $w \cdot ch V = ch V$ since V is integrable.

So

$$w(K \cdot ch V) = \det w \cdot (K\, ch V).$$

Now, $K \cdot ch V = \sum c_\psi e^{\psi+\rho}$.

So

$$\det w \cdot \sum c_\psi e^{\psi+\rho} = w\left(\sum c_\psi e^{\psi+\rho}\right) = \sum c_{w^{-1}\circ\psi} e^{\psi+\rho}$$

$$w^{-1}\circ\psi = w^{-1}(\psi+\rho)-\rho$$

$\Rightarrow$ $c_{w^{-1}\circ\psi} = \det(w) \cdot c_\psi$. So $c_{w\circ\chi} = \det(w)\, c_\chi$, and lemma is proved.

Lemma 9 Let $D = \{\psi \mid c_{\psi-\rho} \neq 0\}$.
Then $D = W(\chi+\rho)$ (one orbit).

Pf. It's clear that
$W(\chi+\rho) \subset D$ by the previous Lemma 8.
We want to show that that's it.
Assume not.
It's clear that D is W-invariant
since ch V is W-invariant (as shown above
But we also proved in Lemma 1(2) ~~that~~ that D must
have an element in P_+: $\exists \beta$, $\beta \in D$,
$\beta \in P_+$, $\beta \notin W(\chi+\rho)$. ~~Assume $b \in W(\chi+\rho)$ (so $b \neq \chi+\rho$~~
~~Since the only elt of $W(\chi+\rho)$ in P_+ is~~
~~$\chi+\rho$). Then~~

$$b-\rho \in D(\chi) \iff b \in D(\chi+\rho).$$

But we proved in Lemma 5 ~~that~~ $(b,b)-(\chi+\rho, \chi+\rho)<0$
But we know by Lemma 6 ~~that~~ for ψ that
do occur in the sum $(\psi+\rho, \psi+\rho) = (\chi+\rho, \chi+\rho)$
$\Rightarrow\Leftarrow$.

So we get (2) of the theorem (using Prop 3). i.e. that $w \mapsto w(\chi+\rho)$ is bijective.
To prove (1), just note that (2) holds for
any highest weight module.