Lecture 18. Fix a simple f.d. Lie alg. $\mathfrak{g}$ of rank r.

Last time we defined the ring

$$R = \varinjlim_{\mu \in \mathfrak{h}^*} e^{\mu}\,\mathbb{C}[[e^{-\alpha_1},\dots,e^{-\alpha_r}]]$$

Then the formal character of every module in category $\mathcal{O}$ is in R, and

$$\mathrm{ch}\, M_\lambda = \sum_{\beta \in Q_+} p(\beta)\, e^{\mu-\beta} = \frac{e^{\mu}}{\prod_{\alpha>0}(1-e^{-\alpha})}$$

$p(\beta)$ – number of partitions of β into positive roots

This shows that $\{\mathrm{ch}\, M_\lambda\}$ are a "topological basis" of R (in fact, $\{\mathrm{ch}\, M_\lambda,\ \lambda \in \mu - Q_+\}$, is a topological basis of $e^{\mu}\,\mathbb{C}[[e^{-\alpha_1},\dots,e^{-\alpha_r}]]$). In other words, any $f \in R$ is $f = \sum b_\lambda\, \mathrm{ch}\, M_\lambda$ (infinite sum).

Ex. For $\mathfrak{sl}_2$, $\mathrm{ch}\, M_\lambda = e^{\lambda} + e^{\lambda-\alpha} + e^{\lambda-2\alpha} + \dots$

$$= \frac{e^{\lambda}}{1-e^{-\alpha}} = \frac{x^{\lambda}}{1-x^{-2}} \quad \text{if } e^{\alpha} = x^2$$

$\lambda \in \mathbb{Z}_+$: $\mathrm{ch}\, L_\lambda = \dfrac{x^{\lambda} - x^{-\lambda-2}}{1-x^{-2}} = \dfrac{x^{\lambda+1} - x^{-\lambda-1}}{x - x^{-1}} = x^{\lambda} + x^{\lambda-2} + \dots + x^{-\lambda}$

Prop. 1) $\mathrm{ch}(M_1 \otimes M_2) = \mathrm{ch}\, M_1 \cdot \mathrm{ch}\, M_2$

2) Suppose $0 \to N \to M \to M/N \to 0$ is an exact sequence in $\mathcal{O}$. Then $\mathrm{ch}\, M = \mathrm{ch}\, N + \mathrm{ch}\, M/N$.

Proof 1) $(M_1 \otimes M_2)[\mu] = \bigoplus_{\mu_1+\mu_2=\mu} M_1[\mu_1] \otimes M_2[\mu_2]$

2) $0 \to N[\mu] \to M[\mu] \to (M/N)[\mu] \to 0$ is an exact sequence, so $\dim M[\mu] = \dim N[\mu] + \dim (M/N)[\mu]$

<u>Remark.</u> Character does not define the module, e.g. for $sl(2)$ $M(0) \cong M(-2) \oplus \mathbb{C}$ but they have the same character

Now we want to generalize this to Kac-Moody algebras. This is somewhat tricky since in the affine case, weight spaces in the usual sense are infinite dimensional. (for $\hat{sl}_2$, $ht^n v$ all have the same weight) We need to extend Cartan subalgebra to deal with it.

<u>Def.</u> Let A be a generalized Cartan matrix$^{(r \times r)}$, $\mathfrak{g}(A)$ - the corresponding Kac-Moody algebra. We define

$$\mathfrak{g}_{ext}(A) = \mathfrak{g}(A) \oplus \mathbb{C}D_1 \oplus \cdots \oplus \mathbb{C}D_r$$

where $[D_i, (e,h,f)_j] = 0$ if $i \neq j$

$$[D_i, e_i] = e_i, \quad [D_i, f_i] = -f_i, \quad [D_i, h_i] = 0.$$

We have $\mathfrak{g}_{ext}(A) = \mathfrak{h}_{ext} \oplus n_+ \oplus n_-$,

$\mathfrak{h}_{ext} = \mathfrak{h} \oplus \mathbb{C}D_1 \oplus \cdots \oplus \mathbb{C}D_r$, of dimension $2r$

Now α_i can be regarded as functionals in $\mathfrak{h}^*_{ext}$, as follows: $\alpha_i(D_j) = \delta_{ij}$

Then $[h, a] = \alpha(h)a$, $a \in \mathfrak{g}_\alpha$, $h \in \mathfrak{h}_{ext}$.
Now let us construct a basis of $\mathfrak{h}^*_{ext}$.
(different from the dual basis to D_i, h_i.
Set $F = Q \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_r$

$P = \mathfrak{h}^* \oplus F$ (space of weights)
with basis $\underset{=h_1^*}{\gamma_1}, \ldots, \underset{=h_r^*}{\gamma_r}, \alpha_1, \ldots, \alpha_r$ where γ_i are a dual basis to h_i of $\mathfrak{h}^*$. We can set $\gamma_i(D_j) = 0$, then we get a map $\varphi: P \to \mathfrak{h}^*_{ext}$. The matrix of inner products of γ_i, α_i with h_i, D_i is

$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$, so φ is an identification, and we can think of γ_i, α_i as elements of $\mathfrak{h}^*_{ext}$.

Now we can define the above notions: Category $\mathcal{O}$, Verma modules M_λ, irreducibles L_λ, characters, with the same properties as above, and we no longer have inf dimensional weight spaces. Note that this is different from the above even in the special case of simple Lie algebras, but it is equivalent; let us explain the connection. For $\chi \in \mathfrak{h}^*$, let $\mathcal{O}_\chi$ be the category of modules in $\mathcal{O}$ with weights from $\chi + F \subset \mathfrak{h}^* \oplus F$ (i.e. $\chi - k_1\alpha_1 - k_2\alpha_2 - \cdots - k_r\alpha_r$)

It's easy to see that any $M \in \mathcal{O}$ is a direct sum of modules in $\mathcal{O}_\chi$ for some χ, so $\mathcal{O} = \bigoplus_{\chi \in \mathfrak{h}^*} \mathcal{O}_\chi$

<u>Claim.</u>

If $\chi_1 - \chi_2$ is a linear combination of $\bar{\alpha}_i$ (images of α_i in $\mathfrak{h}^*$) then

$\mathcal{O}_{\chi_1} \cong \mathcal{O}_{\chi_2}$

<u>Pf.</u> In Feigin - Zelevinsky paper.

For $\mathfrak{g}(A)$ simple, $\bar{\alpha}_i$ are linearly independent, so $\mathcal{O}_\chi$ are all the same (as usual category $\mathcal{O}$ for $\mathfrak{g}$). For $\mathfrak{g}(A)$ affine KM algebra, there is one essential parameter, which is the level of representations.

<u>Lemma.</u> 1) The center Z of $\mathfrak{g}(A)$ is

$\{\sum \beta_i h_i,\ \beta_i \in \mathbb{C}; \sum \beta_i a_{ij} = 0\}$. So $\dim Z = \dim \operatorname{Ker} A$

2) If A is a gen. Cartan matrix then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$

3) If A is an indecomposable symmetrizable matrix, then any graded proper ideal in $\mathfrak{g}(A)$ is

contained in the center. (except $A=(0)$, 1×1 matrix)

<u>Proof</u> 1) $X\in Z \Rightarrow \mathbb{C}X$ is an ideal.

May assume X is homogeneous. then $X\in\mathfrak{h}$, so $Z\subset\mathfrak{h}$. If $\sum\beta_i h_i$ central, then $\left[\sum\beta_i h_i, e_j\right] = \sum\beta_i a_{ij}=0$. Conversely, if this is true then $\sum\beta_i h_i$ is central

2) For gen. Cartan matrix, e_i, f_i, h_i are commutators (in fact, suffices that $a_{ii}\neq 0$)

3) If $I\neq 0$ is a graded ideal, $I\cap\mathfrak{h}\neq 0$..

$I = I_+\oplus I_0\oplus I_-$, $I_0\subset\mathfrak{h}$, I_+, I_- in pos/neg degree

Suppose $I\not\subset Z$. Then I_+ or $I_-\neq 0$. Otherwise $I\ni \mathfrak{h}\notin Z$ and $[h,e_i]\neq 0$ for some i, $\Rightarrow\!\Leftarrow$.

Assume $I_+\neq 0$. let $a\in I_+^{[\alpha]}$, $J=\langle a\rangle$.

Then J is a graded ideal. So $J\cap\mathfrak{h}\neq 0$.

So $\exists\, x\in U(\mathfrak{g}(A))$, $x\circ a \underset{\neq 0}{\in}\mathfrak{h}$. We may assume that x is homogeneous of degree $-\alpha$.

We may assume $x = f_{i_1}\cdots f_{i_n}\cdot e_{j_1}\cdots e_{j_m}$.

then $x\cdot a \sim h_i$, so $e_i, h_i\in I$ for some $i\in I$. Hence $f_i\in I$.

Then if $a_{ij}\neq 0$, so are e_j, f_j, h_j! (as $[h_i, f_j] = -a_{ij}f_j$, $[h_i,e_j]=a_{ij}e_j$)

So all generators are in I and $I=\mathfrak{g}(A)$.

Now we want to introduce invariant form and Casimir operator for $\mathfrak{g}(A)$.

Let A be an indecomposable complex matrix. We want to classify symmetric forms $B : \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ such that B has degree 0, i.e. $\mathfrak{g}_\alpha \times \mathfrak{g}_\beta \to \mathbb{C}$ is zero unless $\alpha + \beta = 0$, and B is invariant (and similarly for $\tilde{\mathfrak{g}}(A)$) (we don't ask for B to be nondegenerate).

We denote:

$(e_i, f_i) = d_i$

Let us assume $d_i \neq 0$ (otherwise form is "too degenerate"). We get

$$(h_i, h_j) = (h_i, [e_j, f_j]) = ([h_i, e_j], f_j)$$

$= a_{ij} d_j \Rightarrow$ we need $a_{ij} d_j = a_{ji} d_i$

This means that we need the matrix A to be symmetrizable ($\exists D$ - diagonal, nondeg s.t. AD is symmetric).

<u>Exercise</u>. If A is indecomposable symmetrizable, D is unique up to a common factor (without assuming nondeg. of D).

Now suppose that A is symmetrizable and let us see when the form $(,)$ exists. -

Prop. If A is indec. symmetrizable and the form $(\,,\,)$ exists, it is unique up to scaling.

Pf. Supp. $B(\,,\,)$ is such a form. We can regard B as a homomorphism
$$B^\vee : \mathfrak{g} \to \mathfrak{g}^*$$
(where $\mathfrak{g}^*$ is the restricted dual)

This is a $\mathfrak{g}$-module hom, since B is invariant. Set $(e_i, f_i) = d_i$ (unique up to scaling). Then

1) $B(h_i) \in \mathfrak{h}^*$ $\quad B(h_i)(h_j) = a_{ij}\, d_j$

2) $B(e_i) \in \mathfrak{g}^*_{-\alpha_i} \Rightarrow B(e_i) = c_i f_i^*$

$B(e_i)(f_i) = d_i$ so $B(e_i) = d_i f_i^*$

Similarly $B(f_i) = d_i e_i^*$.

But then B extends in a unique way since e_i, h_i, f_i generate $\mathfrak{g}$ as a $\mathfrak{g}$-module.

Now let us talk about the existence of B.

Prop. If A is an indecomposable symmetrizable then the form on $\mathfrak{g}(A)$ and $\bar{\mathfrak{g}}(A)$ exists.

Proof. We can define B by induction:

$$B([e_{i_1},[e_{i_2}\cdots[e_{i_{m-1}},e_{i_m}]]],[f_{j_1}\cdots[f_{j_{m-1}},f_{j_m}]])$$

$$=-B([e_{i_2}\cdots e_{i_m}],[e_{i_1},[f_{j_1}\cdots f_{j_m}]])=\cdots$$

One can check it is consistent (in Feigin Zelevinsky paper, p. 51). We will skip it.

Prop. 1) The kernel of $\widetilde{\mathfrak{g}}(A)\to\mathfrak{g}(A)$ is in $\operatorname{Ker}B$, and the form on $\mathfrak{g}(A)$ is the descended form.

2) $\operatorname{Ker}B|_{\mathfrak{g}(A)}=Z$ (the center).

Proof. 1) Follows easily by uniqueness as we can pull back the form from $\mathfrak{g}(A)$

2) Let $\operatorname{Ker}B|_{\mathfrak{g}(A)}=J$.

this is a graded ideal.

Also $J\neq\mathfrak{g}(A)$. So by the lemma above $\operatorname{Ker}B\subset Z$. But also $Z\subset\operatorname{Ker}B$ by Lemma above: $(x,h_j)=\sum\beta_i a_{ij}d_j=0$.

$x\in Z$
$x=\sum\beta_i h_i$

The proposition is proved.

This will allow us to construct a nondegenerate invariant form on ~~$\mathfrak{g}_{ext}$~~.

We have $\gamma: F\to\mathfrak{h}$,

$\gamma(\alpha_i)=d_i^{-1}h_i=h_{\alpha_i}$

$\varphi: P\times P\to\mathbb{C}$ $\quad P=\mathfrak{h}^*\oplus F$

$(\varphi,\alpha)(\psi,\beta)=\varphi(h_\beta)+\psi(h_\alpha)+(h_\alpha,h_\beta)$

One can extend by linearity:
$\gamma(\alpha) = h_\alpha$.

Claim: $(h_\alpha, h) = \bar{\alpha}(h)$, $h \in \mathfrak{h}$, $\bar{\alpha}$ - image of α in L,

Proof $(h_{\alpha_i}, h_j) = (d_i^{-1} h_i, h_j) = d_i^{-1} a_{ij} d_j = d_i^{-1} a_{ji} d_i$
$= a_{ji}$; also $\alpha_i(h_j) = a_{ji}$

Prop If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then
$[x, y] = (x, y) h_\alpha$ $\quad (\alpha = \sum k_i \alpha_i,\ k_i \geq 0)$

~~Pf~~ Proof by induction in $|\alpha| = \sum k_i$,

1) $|\alpha| = 1$, $\alpha = \alpha_i$ $\quad [e_i, f_i] = (e_i, f_i) h_{\alpha_i} = d_i d_i^{-1} h_i$
- OK.

2) $|\alpha| > 1$. $X = [e_i, x']$,
where $x' \in \mathfrak{g}_{\alpha - \alpha_i}$, $y \in \mathfrak{g}_{-\alpha + \alpha_i}$, where $\alpha - \alpha_i$ is a "positive root".

Then
$$[[e_i, x'], y] = [[e_i, y], x'] + [e_i, [x', y]] \overset{\text{ind. ass}}{=} -([e_i, y], x') h_{\alpha - \alpha_i}$$
$$+ (e_i, [x', y]) h_i = ([e_i, x'], y)(h_{\alpha - \alpha_i} + h_{\alpha_i})$$
$$= (x, y) h_\alpha.$$

Corollary. $\mathfrak{g}(A)$ is a nondegenerate Lie algebra