# Lecture 17

$\mathfrak{g}(A') \cong \mathfrak{g}(A)$ if $A' = \sigma A \sigma^{-1}$, $\sigma$ perm.
$\mathfrak{g}(A_1 \oplus A_2) = \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$

Last time we defined, for any complex matrix $A$, the contragredient Lie algebra $\mathfrak{g}(A)$. It can be shown that $\mathfrak{g}(A)$ is finite dimensional iff $A$ is the Cartan matrix of a semisimple Lie algebra (in which case $\mathfrak{g}(A)$ is that Lie algebra). So $I = I_+ \oplus I_-$ is generated by Serre relations.

But if $A$ is generic (in the Weil sense) one can show that $I = 0$, and $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A) = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, and $\mathfrak{n}_+, \mathfrak{n}_-$ are free.

<u>Def.</u> $A$ is called a generalized Cartan matrix if

1. $a_{ii} = 2$
2. $a_{ij} \leq 0$, and $a_{ij} = 0 \iff a_{ji} = 0$ $(i \neq j)$
3. $A$ is symmetrizable, i.e. $\exists$ diagonal matrix $D$ with positive entries s.t. $(DA)^T = DA$.

<u>Remark</u> A Cartan matrix is always a generalized Cartan matrix, and a generalized Cartan matrix is Cartan iff $DA$ is positive definite.

Indeed, a Cartan matrix is symmetrizable since the Dynkin diagram of a simple Lie algebra is a tree.

Ex. $\begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix}$, $m \geq 1$ are generalized Cartan

$m = 1$ $\mathfrak{sl}_3$ $(A_2)$

$m = 2$ $\mathfrak{sp}_4$ $(B_2 = C_2)$

$m = 3$ $\mathfrak{g}_2$

$m \geq 4$ — generalized Cartan

Def. A Kac-Moody algebra attached to a generalized Cartan matrix $A$ is the Lie algebra $\mathfrak{g}(A)$.

Theorem. (Gabber-Kac) For a Kac-Moody algebra $\mathfrak{g}(A)$, the ideal $I \subset \tilde{\mathfrak{g}}(A)$ is generated by the Serre relations $(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0$, $(\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0$.

Def. A Kac-Moody algebra $\mathfrak{g}(A)$ attached to an indecomposable $A$ is called affine if $DA \geq 0$, but $DA \not> 0$ (i.e., $\det A = 0$).

The case when $DA$ is indefinite corresponds to "hyperbolic" KM algebras (they are big, e.g. have exponential growth)

Proof of Gabber - Kac thm. We'll prove only one direction, the other is more difficult. We'll show that Serre relations hold in $\mathfrak{g}(A)$, i.e. $(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j \in I$

Indeed, we just need to show that $\mathrm{ad}\, e_k$ acts on $(\mathrm{ad}\, f_i)^{1-a_{ij}} f_j$ by zero (the $e$-relation is handled similarly)

Case 1: $k \neq i, j$ clear

Case 2: $k = j$. Then $\mathrm{ad}\, e_k (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j$

$= (\mathrm{ad} f_i)^{1-a_{ij}} [e_j, f_j] = (\mathrm{ad}\, f_i)^{1-a_{ij}} h_j = 0$

if $a_{ij} \neq 0$. But for $a_{ij} = 0$ we also get $0$ since $[f_i, h_j] = \alpha_i(h_j) f_i = a_{ji} f_i = 0$.

Case 3: $k = i$. Consider the $(\mathfrak{sl}_2)_i$-module generated over $\langle e_i, f_i, h_i \rangle$ by $f_j$.

We have $[e_i, f_j] = 0$, $[h_i, f_j] = -a_{ij} f_j$, so $M$ is a highest weight module with highest weight $-a_{ij} = m \geq 0$

In this module $(\mathrm{ad}\, f_i)^{m+1} f_j$ is a singular vector, so we get zero. ∎

# Untwisted affine Lie algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra / $\mathbb{C}$.

$L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$, $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$.

**Theorem.** $\hat{\mathfrak{g}}$ is an affine Kac-Moody algebra with an affine Cartan matrix $\hat{A}$ which is indecomposable and contains $A$ as a diagonal block:

$$\hat{A} = \begin{pmatrix} 2 & - \\ | & A \end{pmatrix}$$

*Proof.* Let $r = \operatorname{rank}(\mathfrak{g})$. Let us define $h_i, e_i, f_i$, $i = 0, \ldots, r$ to be the corresponding ~~elements for~~ $\mathfrak{g}$ if $i > 0$, and $e_0 = f_\theta t$, $f_0 = e_\theta t^{-1}$, $h_0 = K - h_\theta$, where $e_\theta, f_\theta, h_\theta$ is the $\mathfrak{sl}_2$-triple corresponding to the maximal root $\theta$. Clearly, $\hat{\mathfrak{g}}$ is generated by $e_i, f_i, h_i$, $i = 0, \ldots, r$.

We have

$$[h_0, f_0] = [h_0, e_\theta t] = [K - h_\theta, e_\theta t]$$

$$= -[h_\theta, e_\theta] t = -\theta(h_\theta) e_\theta t = -2 e_\theta t = -2 f_0$$

similarly $[h_0, e_0] = 2e_0$.

Also $[e_0, f_0] = [f_\theta t, e_\theta t^{-1}] = -h_\theta + K = h_0$.

Also $[h_0, h_i] = 0$,

$[h_0, e_i] = [K - h_\theta, e_i] = -(\alpha_i, \theta) e_i$ $i > 0$

$[h_0, f_i] = (\alpha_i, \theta) f_i$, $i > 0$ so $a_{0i} = -(\alpha_i, \theta)$.

(where the form is normalized so that $(\theta, \theta) = 2$). It is clear that this is an integer.

$[e_0, f_i] = [f_\theta t, f_i] = 0$ since $\theta$ is maximal. Similarly $[f_0, e_i] = 0$.

$[h_i, e_0] = [h_i, f_\theta t] = -\theta(h_i) e_0 = -(\alpha_i^\vee, \theta) e_0$

$[h_i, f_0] = +(\alpha_i^\vee, \theta) f_0$, so $a_{i0} = -(\theta, \alpha_i^\vee)$.

So the relations hold. let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}$

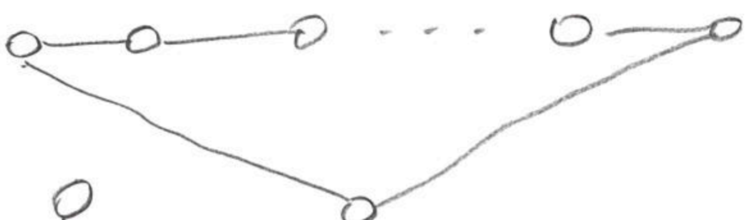
let $\alpha_0 = \delta - \theta$, $\in \hat{\mathfrak{h}}^*$ where $\delta(K) = 1$, $\delta|_{\mathfrak{h}} = 0$; then we see that $[h, e_0] = \alpha_0(h) e_0$, $[h, f_0] = -\alpha_0(h) f_0$. So if we define $Q = \bigoplus_{i=0}^{r} \mathbb{Z}\alpha_i$ then $\hat{\mathfrak{g}}$ is $Q$-graded, with $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$, $\deg(h_i) = 0$

So it remains to show that $\hat{\mathfrak{g}}$ does not have nonzero graded ideals which have zero intersection with $\hat{\mathfrak{h}}$

For this purpose, it's enough to show that $Lg = g[t,t^{-1}]$ has no nontrivial graded ideals.

But if $J \neq 0$ is such ideal, then it contains an element $at^m$, $a \neq 0$, so it generates the whole $Lg$.

<u>Examples</u>: 1) $A_{n-1} = sl_n$. $\theta = (1, 0, \dots, -1)$ so since $\alpha_1 = (1,-1,0,\dots,0)$, ..., $\alpha_{n-1} = (0,\dots,0,1,-1)$ we have $(\theta, \alpha_i) = 0$ except $(\theta,\alpha_1) = 1$, $(\theta, \alpha_{n-1}) = 1$, so we get $a_{01} = -1$, $a_{0n-1} = -1$, and we get Dynkin diagram

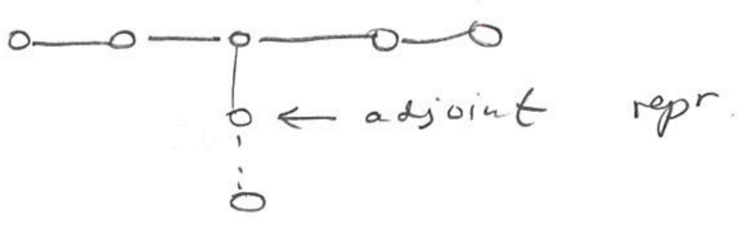


②  $so(2n) = D_n$, $n \geq 4$

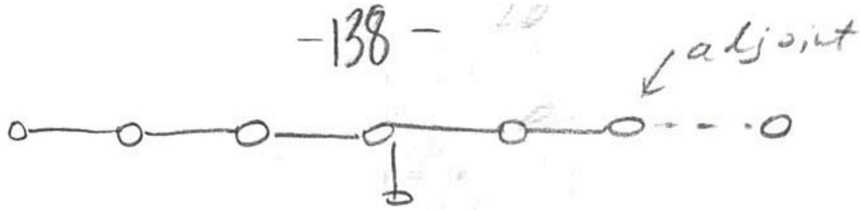
adjoint repr $g = \Lambda^2 V \;\; \overset{\shortparallel}{so}(V)$

(if adjoint repr is fundamental, attach new vertex to this vertex

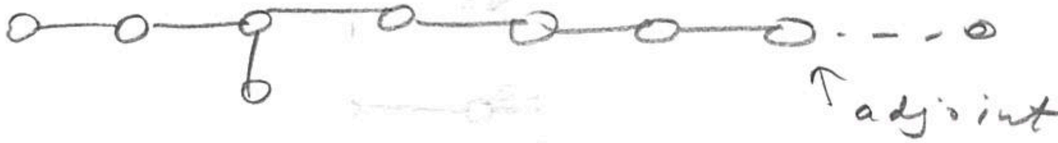
③ $E_6$:



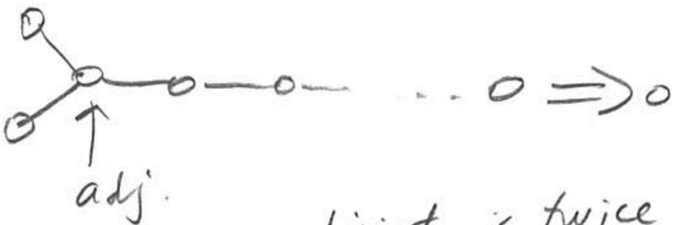
④ $E_7$




⑤ $E_8$

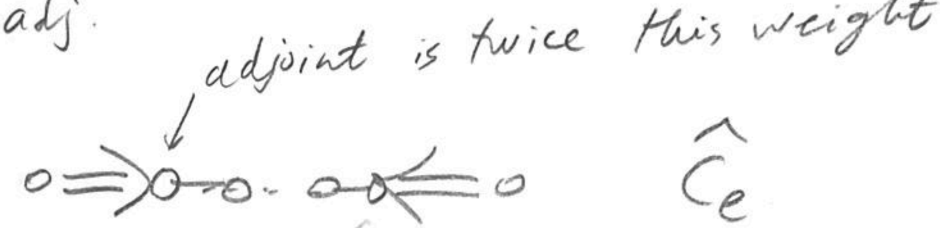



⑥



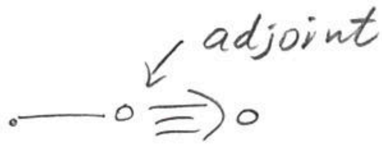

$\hat{B}_\ell$

⑦

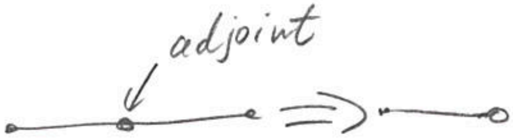



$\hat{C}_\ell$

⑧ $G_2$




⑨




$F_4$

We have $[K, x] = 0$ which implies that we have $\hat{A}$ degenerate. But $\hat{A}$ is nonnegative and kernel is 1-dimensional.

Affine algebras are the most interesting KM algebras since they have two definitions — loop and Kac-Moody. Their interesting properties come from interaction of these two definitions

<u>Def.</u> Category $\mathcal{O}$ over $\mathfrak{g}$ (semisimple)

$M \in \mathcal{O}$ if

1) $M$ is $\mathfrak{h}$-diagonalizable, + weight spaces finite dim.
2) All weights of $M$ are contained in the finite union of $D(\lambda)$ $= \{\lambda - n_1\alpha_1 - \cdots - n_r\alpha_r\}$.

e.g. Verma modules $M_\lambda$, irreducible modules $L_\lambda$ their sums are in $\mathcal{O}$.

Formal character of $M \in \mathcal{O}$

$$ch\, M = \sum e^{\mu} \dim M[\mu].$$

lie in $R = \{\sum a_\mu e^\mu\}$ s.t. this is supported on finite union of the sets $D(\lambda)$.