Lecture 15

Now let us specialize to an example.
Let $\mathfrak{a}$ be a simple Lie algebra.
Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$, and $\mathfrak{p} = \mathfrak{a}_{diag} \subset \mathfrak{g}$.
We endow $\mathfrak{a}$ with the standard form $(\ ,\ )$ such that $(\alpha, \alpha) = 2$ for long roots.
Let V', V'' be admissible representations of $\mathfrak{a}$ (i.e. such that $\forall v\ \forall a \in \mathfrak{a}$ $\exists N\ \forall n \geq N$ $a(n)v = 0$) of levels k', k''. Let $V = V' \otimes V''$.
It is a projective representation of $L\mathfrak{g}$, and it restricts to a repr. of $\hat{\mathfrak{p}}$ of level $k' + k''$. So we have

$$c_{\mathfrak{g}} = c'_{\mathfrak{a}} + c''_{\mathfrak{a}} = \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee}\right) \dim \mathfrak{a}.$$

On the other hand, $c_{\mathfrak{p}} = \dfrac{k'+k''}{k'+k''+h^\vee} \dim \mathfrak{a}$

So
$$c = c_{\mathfrak{g}} - c_{\mathfrak{p}} = \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee} - \frac{k'+k''}{k'+k''+h^\vee}\right) \dim \mathfrak{a}.$$

Example
$\mathfrak{g} = \mathfrak{sl}_2$ $h^\vee = 2$, $k' = 1$, $k'' = 2$.

$$c = \left(\frac{1}{3} + \frac{2}{4} - \frac{3}{5}\right) \cdot 3 = \frac{7}{10} < 1$$

More generally, if $k' = 1$, $k'' = m$,
we get $c = 3\left(\frac{1}{3} + \frac{m}{m+2} - \frac{m+1}{m+3}\right)$

$$= 3\,\frac{(m+2)(m+3) + 3m(m+3) - 3(m+1)(m+2)}{3(m+2)(m+3)}$$

$$= \cancel{3}\,\frac{m^2 + 5m + \cancel{6} + \cancel{3m^2} + \cancel{9m} - \cancel{3m^2} - \cancel{9m} - \cancel{6}}{\cancel{3}(m+2)(m+3)}$$

$$= 1 - \frac{6}{(m+2)(m+3)}.$$

So we are now able to construct unitary representations of Vir with such c (multiplicity spaces in tensor product).

To develop this theory in more detail, we need to derive character formulas for representations of affine Lie algebras. We will do it in the more general setting of Kac-Moody algebras, and this will be one of the main goals of the course.

Kac-Moody algebras

Recall that if $\mathfrak{g}$ is a simple Lie algebra then a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a maximal commutative subalgebra consisting of semisimple elements. It is not unique, but unique up to action of the corresponding group G. If we fix $\mathfrak{h} \subset \mathfrak{g}$, we have an eigenspace decomposition

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $R \subset \mathfrak{h}^*$

(as $\mathfrak{h}$ acts semisimply on $\mathfrak{g}$).

Here for $\alpha \in \Delta$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h x] = \alpha(h) x$
It is known that $\mathfrak{g}_\alpha$ is 1-dimensional. $\forall h \in \mathfrak{h}\}$.

Let us pick $\bar{h} \in \mathfrak{h}$ such that $\alpha(\bar{h})$ are real and $\neq 0$ $\forall \alpha$, and set

$\Delta_\pm = \{\alpha \in \Delta \mid \alpha(\bar{h}) \gtrless 0\}$.

Set $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$, $\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$ - Lie subalgebras

$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ - triangular decomposition, $\Delta = \Delta_+ \cup \Delta_-$ $\Delta_+ \leftrightarrow \mathfrak{n}_+$
$\Delta_- \leftrightarrow \mathfrak{n}_-$

Δ_+ has a "basis" α_i, $i=1,\dots,r$ simple roots
(which is a basis of $\mathfrak{h}^*$):
Any $\alpha \in \Delta_+$ can be written as
$\alpha = \sum n_i \alpha_i$, $n_i \geq 0$, $n_i \in \mathbb{Z}$, in a unique way
Know that: $\alpha+\beta \notin \Delta \cup 0 \Rightarrow$
$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$
If $\alpha, \beta \in \Delta$ s.t. $\alpha+\beta \in \Delta$ then
$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
if $\alpha+\beta=0$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{h}$.
We call e_α the unique up to scale element of $\mathfrak{g}_\alpha$, $\alpha \in \Delta_+$
f_α the unique up to scaling elt of $\mathfrak{g}_\alpha$, $\alpha \in \Delta_-$
In particular: $e_{\alpha_i} = e_i$, $f_{\alpha_i} = f_i$
$h_i = [e_i, f_i] \in \mathfrak{h}$.
We can normalize e_i, f_i so that
$[h_i, e_i] = 2e_i$, $[h_i, f_i] = -2f_i$
(sl_2 - triple)

And we have

Prop. 1) h_i is a basis of $\mathfrak{h}$

2) $\mathfrak{g}$ is generated by h_i, e_i, f_i

(proof is in the f.d. Lie alg. course)

What are the relations:

Prop. $[h_i, h_j] = 0$

$[h_i, e_j] = \alpha_j(h_i)\, e_j$

$[h_i, f_j] = -\alpha_j(h_i) f_j$

$[e_i, f_j] = \delta_{ij} h_i$

This is clear ($\alpha_i - \alpha_j$ is not a root).

But these are not all the relations (except for sl_2).

In other words, if $\tilde{\mathfrak{g}}$ is the Lie algebra with such relations then $\tilde{\mathfrak{g}}$ is infinite dimensional and $\varphi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is not injective.

That's because there are additional relations involving only e_i and f_i called Serre relations. We'll discuss them.

Let us identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using an invariant form.

So $\mathfrak{h} \cong \mathfrak{h}^*$.

$h_i \to \alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$ (where $(\,,\,)$ is the inverse form of $\mathfrak{h}^*$)

$a_{ij} \overset{\text{def}}{=} \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ Cartan matrix

So $[h_i, e_j] = a_{ij} e_j$

$[h_i, f_j] = -a_{ij} f_j$

Properties of $A = (a_{ij})$:

1) $a_{ii} = 2$;

2) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$; $a_{ij} \overset{\in \mathbb{Z}}{\leq} 0$ if $i \neq j$

3) A is indecomposable, i.e. $A \not\cong A_1 \oplus A_2$ after perm. of rows and columns

4) A is "positive definit".

I.e. $\exists$ D diagonal positive such that DA is positive definite.

<u>Thm</u>. A matrix satisfies these properties $\Leftrightarrow$ it is a Cartan matrix of a simple Lie algebra

Such matrices are encoded by Dynkin digrams

The Cartan matrix can be encoded by a Dynkin diagram:

Vertices are α_i

If $a_{ij}=0$, i is not connected to j

If $a_{ij}=-1$, i is connected to j

If $a_{ij}=-2$, $a_{ji}=-1$, $i \Leftarrow j$

If $a_{ij}=-3$, $a_{ji}=-1$, $i \Lleftarrow j$

Note that $a_{ij}\geqslant -3$ since $\det\begin{pmatrix}a_{ii} & a_{ij}\\ a_{ji} & a_{jj}\end{pmatrix}>0$.

$$\det\begin{pmatrix}a_{ii} & a_{ij}\\ a_{ji} & a_{jj}\end{pmatrix} = 4-a_{ij}a_{ji}$$

so $-3\leq a_{ij}\leq -1$

Classification:

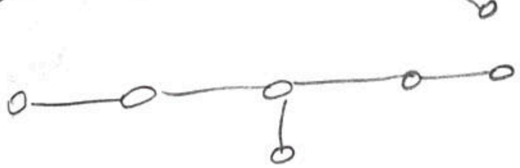
$A_{n-1}=sl_n$, $n\geqslant 2$

$B_n = so(2n+1)$, $n\geqslant 2$

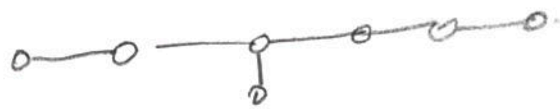
(subscript n is the number of vertices)

$C_n = sp(2n)$, $n\geqslant 3$

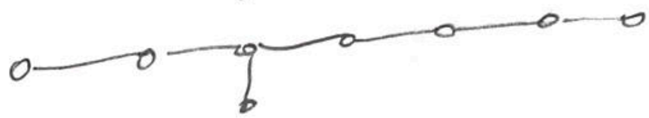
$D_n = so(2n)$, $n\geqslant 4$



E_6



E_7



$E8$

F_4

G_2

Mnemonic rule:

$i\Leftarrow j$ or $i \Lleftarrow j$

$\Rightarrow |\alpha_i|^2<|\alpha_j|^2$

Theorem. Let $i \neq j$. Then in $\mathfrak{g}$

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$$

$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$$

So
$a_{ij} = 0 \Rightarrow [e_i, e_j] = 0$

$a_{ij} = -1 \Rightarrow [e_i [e_i, e_j]] = 0$

$a_{ij} = -2 \quad [e_i [e_i [e_i e_j]]] = 0$

$a_{ij} = -3 \quad [e_i [e_i [e_i [e_i e_j]]]] = 0$

Proof. $[e_i, f_j] = 0$.
So f_j is a highest vector for $(\mathfrak{sl}_2)_i$.
$[h_i, f_j] = -a_{ij} f_j$, so highest weight is $-a_{ij}$.
So $(\operatorname{ad} f_i)^{1-a_{ij}} f_j$, which has weight
$-a_{ij} - 2(1 - a_{ij}) = a_{ij} - 2$, so it is zero.

Theorem. Any simple Lie algebra $\mathfrak{g}$ is $\tilde{\mathfrak{g}}$ / (Serre relations)

Now let's pass to ∞-dimensional case.

Contragredient Lie algebras

Let $A = (a_{ij})$ be an $n \times n$ matrix of complex elements.

Q - free abelian group of rank n,
basis $\alpha_1, \dots, \alpha_n$ (root lattice).

<u>Def.</u> A contragredient Lie algebra corresponding to A is a Lie algebra $\mathfrak{g}$ with generators e_i, f_i, h_i, $i=1,\dots,n$ with relations

1) $[h_i, h_j] = 0$
   $[h_i, e_j] = a_{ij} e_j$
   $[h_i, f_j] = -a_{ij} f_j$
   $[e_i, f_j] = \delta_{ij} h_i$ (possibly and others)

such that

2) $\mathfrak{g}$ is Q-graded

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha \qquad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

$\mathfrak{g}_0 = \langle h_1, \dots, h_m \rangle$

$\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$

$\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$

3) Every nonzero Q-graded ideal has a nonzero intersection with $\mathfrak{h}$.

(If $\mathfrak{g}$ is simple, 3) is automatically satisfied)

Proposition. If A is a complex matrix then there exists a unique contragredient Lie algebra $\mathfrak{g}$ corresponding to A (up to a graded isomorphism).

Notation. Such $\mathfrak{g}$ is denoted by $\mathfrak{g}(A)$.

Proof. Consider the algebra $\tilde{\mathfrak{g}}(A)$ generated by e_i, f_i, h_i with defining relations (1).

Claim. We have

$$\tilde{\mathfrak{g}}(A) = \tilde{n}_- \oplus \mathfrak{h} \oplus \tilde{n}_+$$

where $\mathfrak{h} = \mathrm{span}(h_i)$ (where h_i are a basis) $\tilde{n}_+$ is the free Lie algebra generated by e_i, $\tilde{n}_-$ is the free Lie algebra generated by f_i

Pf of the Claim. Consider space

~~$\tilde{\mathfrak{g}}' = \tilde{n}_- \oplus \mathfrak{h} \oplus \tilde{n}_+$, where $\mathfrak{h} = \oplus \mathbb{C} h_i$, $\tilde{n}_+$ is the free Lie algebra in e_i, $\tilde{n}_-$ is the free Lie algebra in f_i, the commutator of h_i with $\tilde{n}_+$ and $\tilde{n}_-$ is given by (1), the commutator of e_i with $\tilde{n}_-$ is given by (1), and this is extended to an action of $\tilde{n}_+$ on $\tilde{n}_-$~~

Proof of the claim. Let $\tilde{n}_+, \tilde{n}_-, \mathfrak{h}$ be the subalgebras of $\tilde{\mathfrak{g}}(A)$ generated by the e_i, f_i, h_i respectively. It is easy to see by looking at the grading that $\tilde{n}_+ \oplus \mathfrak{h} \oplus \tilde{n}_- \subset \tilde{\mathfrak{g}}(A)$. Also by constructing an action of $\tilde{\mathfrak{g}}(A)$ on $\mathrm{FreeLie}(e_i) \oplus \bigoplus_i \mathbb{C}h_i \oplus \mathrm{FreeLie}(f_i)$, one can see that $\tilde{n}_\pm$ are free and $\mathfrak{h}$ has the basis h_i (this can be seen for generic A, hence always). It remains to show that $\tilde{n}_+ \oplus \mathfrak{h} \oplus \tilde{n}_- = \tilde{\mathfrak{g}}(A)$, i.e. that every commutator which is homogeneous under the Q-grading is either in $\tilde{n}_+$ or in $\mathfrak{h}$ or in $\tilde{n}_-$. But this is clear: first of all, all h_i can be removed using the relations for $[h_i, e_j]$, $[h_i, f_j]$, then if we have $[x_1, [x_2, \ldots [x_{n-1}, x_n]]]$ with $x_k = e_i$ or $x_k = f_i$, we assume that $x_n = f_i$ and take the largest k such that $x_k = e_j$, and use commutation relations to shorten the commutator. The Claim is proved.

Now, we see that $\tilde{\mathfrak{g}}(A)$ is Q-graded and satisfies conditions (1) and (2). It does not, in general, satisfy condition (3), so we define I to be the sum of all Q-graded ideals in $\tilde{\mathfrak{g}}(A)$ that have zero intersection with $\mathfrak{h}$. Then $I = I_+ \oplus I_-$, $I_\pm = I \cap \tilde{\mathfrak{n}}_\pm$, and set $\mathfrak{g} = \tilde{\mathfrak{g}}(A)/I$. Then $\mathfrak{g}$ satisfies conditions (1)-(3). Indeed, if $J \subset \mathfrak{g}$ is a graded ideal violating (3) then $\tilde{J} \subset \tilde{\mathfrak{g}}$ (preimage of J) contains I, $\tilde{J} \neq I$, and $\tilde{J} \cap \mathfrak{h} = 0$. $\Rightarrow\Leftarrow$. So $\mathfrak{g}(A)$ exists.

Now we need to prove that $\mathfrak{g}(A)$ is unique. Let $\mathfrak{g}(A)'$ be another one. We have a map $\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A)'$; and I is killed, so we have surjection $\mathfrak{g}(A) \to \mathfrak{g}(A')$. But Kernel must be zero by cond (III), so this is an isomorphism.