Lecture 14

Sugawara construction

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with an invariant symmetric form $(,)$. Recall that we can define the centrally extended loop algebra $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$, with 2-cocycle $\alpha(a,b) = \mathrm{Res}_{t=0}(a'(t), b(t))\,dt$.

Let M be a $\hat{\mathfrak{g}}$-module with $K=k$.

__Def.__ $k \in \mathbb{C}$ is non-critical if $k(,) + \frac{1}{2}\mathrm{Killing}$ is a nondegenerate form on $\mathfrak{g}$.

__Theorem.__ (Sugawara construction) If k is noncritical and $\forall v \in M$ $\exists N$ $\forall n \geq N$ $\forall a \in \mathfrak{g}$, $at^n v = 0$, then the action of $\hat{\mathfrak{g}}$ on M extends to an action of $\mathrm{Vir} \ltimes \hat{\mathfrak{g}}$, with the Virasoro generators acting by the Sugawara formula:

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{a \in B} : a_m a_{n-m} : ,$$

where : B is an orthonormal basis of $\mathfrak{g}$ under the form $k(\ ,\ ) + \frac{1}{2}$Killing; $a_m = a \cdot t^m$; and $: a_m a_r : = \begin{cases} a_m a_r, & m \le r \\ a_r a_m, & m \ge r. \end{cases}$

Moreover, the Virasoro central charge equals $c = k \sum_{a \in B} (a, a)$.

<u>Proof</u> let $\tilde{L}_n$ be the operators defined by the above formulas. They are well defined because of the conditions on M. Moreover, we claim that

$$[\tilde{L}_n, b_r] = -r\, b_{n+r}$$

Indeed, we have $\sum_{a \in B} a \otimes [ab] + \sum_{a \in B} [ab] \otimes a = 0$ and $\sum_{a \in B} (a, [ab]) = 0$.

So when we commute $\tilde{L}_n$ with b_r:

$$[b_r, L_n] = \lim_{N\to\infty} \frac{1}{2} \sum_a \sum_{|m-\frac{n}{2}|\le N} \Big([b_r, a_m]\, a_{n-m}$$

$$+ a_m [b_r a_{n-m}] \Big)$$

$$= \lim_{N\to\infty} \frac{1}{2} \sum_a \sum_{|m-\frac{n}{2}|\le N} \Big([ba]_{r+m}\, a_{n-m} + a_m [ba]_{r+n-m} \Big)$$

$$+ \frac{1}{2} \sum_a 2kr\,(b,a)\, a_{n+r}$$

$$= \lim_{N\to\infty} \frac{1}{2} \sum_a \sum_{|m-\frac{n}{2}|\le N} \Big([ba]_{r+m}\, a_{n-m} - [ba]_m\, a_{r+n-m} \Big)$$

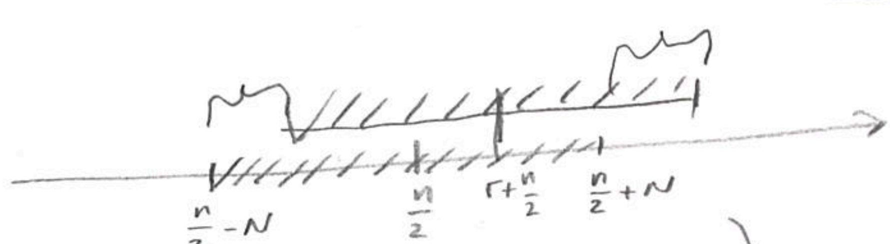
$$\Big(\text{as } \sum_a (a \otimes [ab] + [ab] \otimes a) = 0$$

$$+ \sum_a kr\,(b,a)\, a_{n+r}$$

$$= \lim_{N\to\infty} \frac{1}{2} \sum_a \Big(\sum_{|m-r-\frac{n}{2}|\le N} - \sum_{|m-\frac{n}{2}|\le N} \Big) [ba]_m\, a_{r+n-m}$$

$r \ge 0$

$$+ \sum_a kr\,(b,a)\, a_{n+r}$$



$$\lim_{N\to\infty} \frac{1}{2} \sum_a \Big(-\sum_{\frac{n}{2}-N \le m < \frac{n}{2}-N+r} + \sum_{\frac{n}{2}+N < m \le \frac{n}{2}+N+r} \Big) [ba]_m\, a_{r+n-m}$$

$$+ \sum_a kr\,(b,a)\, a_{n+r}$$

Now, if we apply to any vector v, for $N \gg 0$ the first summand is zero. The second summand is zero after reordering. So we get

$$+\frac{1}{2} r\left[[ba], a\right]_{r+n} + \sum_a kr(b,a)\, a_{n+r}$$

$$\boxed{\begin{aligned} &\underline{\text{Lemma}}: \sum_a [[ba]a] = \cancel{\tfrac{1}{2}}\, \mathrm{Kil}(b,a)\, a \\ &\underline{\text{Proof}}.\ \sum \mathrm{Kil}(b,a)\, a = \sum c_i^*([b[ac_i]])\, a \\ &\qquad = -\sum c_i^*([ba])[ac_i] \\ &\qquad = -\sum [a[ba]] = [[ba]a] \end{aligned}}$$

$$= \frac{1}{2} r\, \mathrm{Killing}(b,a)\, a_{n+r} + kr(b,a)\, a_{n+r} = kr\, b_{n+r}.$$

(use the Lemma)

It remains to show that L_n satisfy the Virasoro relations with appropriate c.

We have $\left[[L_n, L_m] - (n-m)L_{n+m}, a_i\right] = 0$,

so $\left[[L_n, L_m] - (n-m)L_{n+m}, L_0\right] = 0$,

which implies that $[L_n, L_m] - (n-m)L_{n+m} = 0$ for $n+m \neq 0$. (as $[L_0, a_r] = -ra_r$, so $[L_0, L_n] = -nL_n$.

The rest can be proved a direct calculation.

Namely,

$$[L_n, L_{-n}] = \frac{1}{2}\lim_{N\to\infty}\Big(\sum_a \sum_{|m-\frac{n}{2}|\le N} (-m)\, a_{n+m}\, a_{-n-m}$$

$$+\frac{1}{2}\sum_a \sum_{|m-\frac{n}{2}|\le N} (+n+m)\, a_m a_{-m}\Big) =$$

$$\frac{1}{2}\lim_{N\to\infty} \sum_a \Big(\sum_{|m-\frac{3n}{2}|\le N} (-m+n)\, a_m a_{-m}$$

$$+\sum_{|m-\frac{n}{2}|\le N} (m+n)\, a_m a_{-m}\Big).$$

So $[L_n, L_{-n}] - 2nL_0 =$

$$\frac{1}{2}\lim_{N\to\infty} \sum_a \Big(\sum_{\frac{3n}{2}-N\le m<\frac{3m}{2}-N} (+m+n)\, a_m a_{-m} + \sum_{\frac{n}{2}+N\le m\le\frac{3n}{2}+N} (-m+n)\, a_m a_{-m}$$

$$+\sum_{1\le m\le \frac{n}{2}+N} 2nmk(a,a)\Big).$$

So we see that this is a scalar.

$n=1$: $(-N-1+1)(N+1) + N(N+1) = 0.$

$n=2$: $(-N-2+2)(N+2) + (-N-3+2)(N+3) + 2(N+2)(N+1)$

$= -N(N+2) - (N+1)(N+3) + 2(N+2)(N+1)$

$= (N+2) - (N+1) = 1.$ So get $\frac{1}{2}k\sum(a,a)$.

So we get the theorem by the uniqueness of the Virasoro extension.

Now consider the special cases.

1) $\mathfrak{g}$ is an abelian Lie algebra, $(\ ,\ )$ a nondegenerate form. Then the Killing form is 0, so k is noncrit $\Leftrightarrow$ $k \neq 0$, and $a \in B$ should be orthonormal with respect to $k(\ ,\ )$. Hence

$$\sum_{a\in B} k(a,a) = \sum_{a\in B} 1 = \dim \mathfrak{g}.$$ So $c = \dim \mathfrak{g}$.

In particular, for $\dim \mathfrak{g} = 1$ we get $c=1$ (a result from before), and in the general case, we get a tensor product of $\dim \mathfrak{g}$ such representations.

2) $\mathfrak{g}$ a simple Lie algebra. Let us pick $(\ ,\ )$ so that $(\alpha,\alpha)=2$ for long roots for the inverse form (this is the usual normalization). In this case, we have:

Prop. $\mathrm{Killing}(a,b) = 2h^\vee (a,b)$, where $h^\vee = 1 + (\theta,\rho)$ - the dual Coxeter number of $\mathfrak{g}$.

Proof. Let L_λ be the f.d. irreducible repr. of $\mathfrak{g}$ with highest weight $\lambda \in P_+$. Let $C = \sum_{a \in B} a^2$, where B is an orthonormal basis of $\mathfrak{g}$ under $(\,,\,)$. Then

$Tr_{L_\lambda}(C) =$

$\sum_{a \in B} Tr_{L_\lambda}(a^2) = \gamma_\lambda \cdot \dim L_\lambda$, where $\gamma_\lambda = C|_{V_\lambda}$.

So if $L_\lambda = \mathfrak{g}$, we get

$$\sum_{a \in B} Tr_{\mathfrak{g}} (ad(a))^2 = \gamma_\theta \cdot \dim \mathfrak{g} = \gamma_\theta \sum_a (a,a)$$

$$\parallel$$

$$\sum_a \text{Killing}(a,a).$$

Thus $\text{Killing}(a,a) = \gamma_\theta \cdot (a,a)$ and it remains to show that $\gamma_\theta = 2h^\vee$.

But $\gamma_\lambda = (\lambda, \lambda + 2\rho)$, so $\gamma_\theta = (\theta, \theta + 2\rho)$

$= 2 + 2(\theta, \rho) = 2h^\vee$.

Table:

A_{n-1}	n
B_n	$2n-1$
C_n	$n+1$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

So we get

Thm. The Sugawara construction for simple $\mathfrak{g}$ defines a representation of Vir by

$$L_n = \frac{1}{2(k+h^\vee)} \sum :a_m a_{n-m}:$$ where a_i are orthonormal under $(\,,\,)$

with $$c = \frac{k \dim \mathfrak{g}}{k+h^\vee}$$

Corollary. At $k=-h^\vee$ (critical level)

$$T_n \stackrel{\text{def}}{=} \frac{1}{2} \sum :a_m a_{n-m}:$$

commute with $\hat{\mathfrak{g}}$ and each other (i.e. are "central")

So for critical level a highest weight module does not have to be graded; can set T_i, $i>0$ to any numbers τ_i.

Also it is easy to see that the Sugawara construction preserves unitarity. So if V is a unitary representation of $\hat{\mathfrak{g}}$, it is a unitary representation of Vir.

But typically $\dfrac{k \dim \mathfrak{g}}{k+g} \geq 1$

(k must be a nonnegative integer), so it is not too interesting.

The question is, can we get Vir-repr with $c<1$ from this construction.

The answer is yes, but we need to consider a more sophisticated construction called the coset construction.

Suppose $\mathfrak{g} \supset \mathfrak{p}$ are two reductive Lie algebras, and suppose $\mathfrak{g}$ has a form $(\,,\,)$. Let M be a $\hat{\mathfrak{g}}$ module as above. Then we have two actions of Vir on M: $L_i^{\mathfrak{g}}$ and $L_i^{\mathfrak{p}}$ (in the nondegenerate situation)

Theorem. $L_i = L_i^{\mathfrak{g}} - L_i^{\mathfrak{p}}$ defines a Vir action on M with $c = c_{\mathfrak{g}} - c_{\mathfrak{p}}$, and $[L_n, L_k^{\mathfrak{p}}] = 0$.

Proof. $\forall p \in \mathfrak{p}$ $[L_n, p_k] = [L_n^{\mathfrak{g}}, p_k] - [L_n^{\mathfrak{p}}, p_k]$

$= -k p_{n+k} + k p_{n+k} = 0$

So $[L_n, L_k^{\mathfrak{p}}] = 0$.

Thus,

$[L_n, L_m] = [L_n^{\mathfrak{g}} - L_n^{\mathfrak{p}}, L_m^{\mathfrak{g}} - L_m^{\mathfrak{p}}] = [L_n, L_m^{\mathfrak{g}}] =$

$= [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{\mathfrak{g}}]$

$= [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{p}}, L_m^{\mathfrak{p}}]$

$= (n-m)(L_{n+m}^{\mathfrak{g}} - L_{n+m}^{\mathfrak{p}}) + \frac{n^3-n}{12} \delta_{n,-m} (c_{\mathfrak{g}} - c_{\mathfrak{p}})$ ∎