Lecture 12

Recall that the Verma module M_λ, $\lambda = (c, h)$ over Vir has a contravariant form $(,)$, also called Shapovalov form. It has the contravariance property

$$(L_n v, w) = (v, L_{-n} w)$$

and is symmetric.

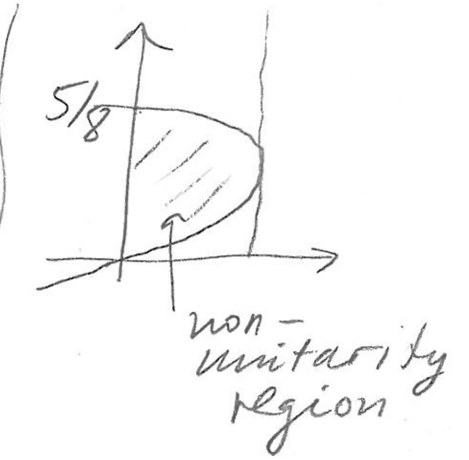
Also, different degrees are orthogonal with respect to this form. Let us consider the determinant polynomial of this form in degree n, $\det_n(c, h)$. As we remember, it is uniquely determined up to scaling.

Prop. (proved before). $\det_n(c, h) = 0$ $\Leftrightarrow$ there is a $\neq 0$ singular vector of degree $\leq n$. Thus, $M_{c,h}$ is irreducible $\Leftrightarrow$ $\det_n(c, h) \neq 0$ $\forall n$.

Corollary $\det_m(c, h) = 0 \Rightarrow \det_{m+1}(c, h) = 0$, so if $X_m = \{\det_m = 0\}$ then $X_m \subseteq X_{m+1}$.

(in fact, we'll see that $\det_{m+1}$ is divisible but $\det_m$)

Ex. $\det_1 = 2h$, $\det_2 = 2h(16h^2 + 2hc - 10h + c)$



Prop. $M(c,h)$ is unitary

$\Rightarrow \det_m(c,h) > 0 \quad \forall m$

(if we use the Hermitian form)

Theorem. Fix c. Then

$$\det_m(c,h) = K_m h^{\sum_{1\le r,s\le m,\ rs\le m} p(m-rs)} + \text{lower terms}$$

where $K_m \neq 0$.

$$K_m = \left(\prod_{1\le r,s\le m} (2r)^s \cdot s!\right)^{m(r,s)}$$

$m(r,s) = p(n-rs) - p(n-r(s+1))$ – the number of partitions of n containing s copies of r

Proof. This follows from a general theorem before (we study asymptotics for fixed c and $h \to \infty$, which corresponds to the "Heisenberg" limit with $c=0$ and h finite). So we just need to compute the degree with respect to h. Suppose we have a partition λ of m:

$m = k_1 + 2k_2 + \cdots + \ell k_\ell$, $k_i = k_i(\lambda)$ multiplicity of i

Then we have monomial $L_{-r}^{k_r} \cdots L_{-1}^{k_1}$ which contributes $k_1 + \cdots + k_\ell$ to the degree. So the degree is

$$D_n = \sum_{\lambda \vdash n} \sum_i k_i(\lambda).$$

Claim. This can be written as

$$\sum_{r,s:\ 1\le r,s\le m} s\, m(r,s)$$

Indeed, we have $m(r,s)$ partitions with r occuring s times, so total number of times r occurs is $\sum_{1\le s\le m} s\,m(r,s)$.

<u>Lemma</u> $\sum_{1\le s\cdot r\le m} s\,m(r,s) = \sum_s p(m-rs)$

<u>Proof.</u> $m(r,s) = p(m-rs) - p(m-r(s+1))$, so the result follows.

The theorem is proved.

Now we will discuss zeros of the determinant polynomials.

<u>Thm.</u> (Kac; another proof given by Feigin + Fuchs)

let, for $1 \le r,s \le m$, $rs \le m$

$$h_{r,s}(c) = \frac{1}{48}\left((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}\,(r^2-s^2) - 24rs - 2 + 2c\right)$$

Then

$$\det_m(c,h) = K_m \prod_{\substack{1\le r,s \\ rs\le m}} (h - h_{r,s}(c))^{p(m-rs)}$$

<u>Remark.</u> For each r,s, we have to choose a branch of square root, but the other branch corresponds to switching r and s.

For the proof, we need the following lemma.

<u>Lemma</u>. Let $A(t)$ be a matrix whose entries are polynomial in t such that $\dim \operatorname{Ker} A(0) \geq N$. Then $\det A(t)$ is divisible by t^N.

<u>Proof</u>. Let $v_1, \ldots, v_N$ be linearly indep. vectors in $\operatorname{Ker} A(0)$. Complete them to a basis. Then A written in this basis has the first N columns divisible by t, which implies the statement.

Now we go back to the proof.

<u>Thm</u>. (Kac) (to be proved later). One has $\det_{rs}(h_{r,s}(c), c) = 0$ (in fact, there is a singular vector of degree rs for $h = h_{r,s}(c)$).

Now, using this theorem, we conclude that $\det_m(c,h)$ is divisible by $(h-h_{r,s})^{p(m-rs)}$, as we have a singular vector in degree $\leq rs$, and it generates a Verma submodule, which in degree m has dimension $\geq p(m-rs)$. Since the total degree and the leading term of $\det_m(h,s)$ is known, and it equals $\sum p(m-rs)$

and K_m, we are done.

The formula can be rewritten as:

$$\det_m(c,h) = K_m \prod_{1\le r\le m} (h-h_{r,r}(c)) \prod_{r<s} (h-h_{r,s}(c))(h-h_{s,r}(c))$$

where $h_{r,r}(c)$ is linear in c (defines lines) L_r

and

$(h-h_{r,s}(c))(h-h_{s,r}(c))$ is quadratic

(defines hyperbolas $\Gamma_{r,s}$)

So outside of a countable set of lines and hyperbolas, $M_{c,h}$ is irreducible.

<u>Corollary</u> 1) Let $h\ge 0$, $c\ge 1$. Then

$L_{c,h}$ is unitary

2) Let $h>0$, $c>0$, then $L_{c,h} \cong M_{c,h}$.

<u>Proof</u> It's easy to see that the hyperbolas and lines don't intersect this open region (the functions are positive there).

$$\left(h-\frac{(r-s)^2}{4}\right)^2 + \frac{h^{(c-1)}}{24}(r^2+s^2-2) + \frac{1}{576}(r^2-1)(s^2-1)(c-1)^2$$

$$+\frac{1}{48}(c-1)(r-s)^2(rs+1) = 0 \quad \text{(hyperbolas)}$$

$$h + (r^2-1)(c-1)/24 = 0 \quad \text{(lines)}$$

This means that the second statement holds. Since we already know that $L_{c,h}$ $(=M_{c,h})$ is unitary for some points of this open region, it should also be true for all points by deformation argument, and also on the boundary by continuity (we use that a positive form cannot become nonpositive without passing through degenerate forms, and limit of a positive form is a nonnegative form).

Now we know a lot about unitary representations. One can show all of them should occur above the lines and in between the branches of the hyperbolas (this is where the region $h>0, c>1$ is). (If we cross, determinant $\det_{rs}$ changes sign).

By detailed analysis of these regions, Friedan-Qui-Shenker showed that the only remaining possible points are the following:

$$c(m) = 1 - \frac{6}{(m+2)(m+3)}, \quad m \geq 0$$

$$h_{s,r}(m) = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}, \quad 1 \leq r \leq s \leq m+1$$

We will show that for these points one has indeed unitary repr. They are called "discrete series". We will not prove that these are the only points, but let us derive the following corollaries

<u>Prop.</u> $c=0$, $L_{c,h}$ unitary $\Rightarrow h=0$

<u>Pf</u> Consider the form on the 2-d space $\langle L_{-N}^2 v, L_{-2N} v \rangle$. Determinant is $4N^3h^2(8h-5N)$, so for this to be ≥ 0 we need $h=0$

<u>Prop.</u> $L_{0,h} = M_{0,h} \Leftrightarrow h \neq \frac{m^2-1}{24}$ $m \in \mathbb{Z}$

$L_{1,h} = M_{1,h} \Leftrightarrow h \neq \frac{m^2}{24}$, $m \in \mathbb{Z}$.

<u>Pf.</u> direct computation with the determinant formula

Lecture 13

Affine Lie algebras

Consider the loop algebra $L\mathfrak{gl}_n = \mathfrak{gl}_n[t,t^{-1}]$. It acts on $\mathbb{C}^n[t,t^{-1}]$. This representation has basis $e_i t^k$, where $e_0,\dots,e_{n-1}$ is a basis of $\mathbb{C}^n$. Denote $e_i t^k$ by v_{i-kn}. Then we get an identification of $\mathbb{C}^n[t,t^{-1}]$ with $V=\mathbb{C}^\infty$, the tautological representation of $\mathfrak{gl}_\infty$. Now consider $a(t)=\sum_k a_k t^k$ (finite sums) and we want to express it as an element of $\overline{\mathfrak{a}}_\infty$ (acting on $\mathbb{C}^\infty=\mathbb{C}^n[t,t^{-1}]$). When we do this, we get a block matrix

$$a(t)\longmapsto\begin{pmatrix} \ddots & & & & & \\ & a_0 & a_1 & a_2 & & \\ & a_{-1} & a_0 & a_1 & a_2 & \\ & & a_{-1} & a_0 & a_1 & a_2 \\ & & & a_{-1} & a_0 & a_1 \\ & & & \ddots & & a_0 \\ & & & & & & \ddots \end{pmatrix}$$

Indeed, if we apply this to a block vector $\begin{pmatrix} 0\\0\\v\\0\\0\\0\end{pmatrix}$, we get

$$\begin{pmatrix}\vdots\\ a_1 v\\ a_0 v\\ a_{-1} v\\ \vdots\end{pmatrix}=\sum_k a_k t^k\cdot v$$

Thus we get an embedding $L\mathfrak{gl}_n\hookrightarrow\overline{\mathfrak{a}}_\infty$.

Now recall that $\overline{\mathfrak{a}}_\infty$ carries a central extension define by the 2-cocycle α. of $L\mathfrak{gl}_n$. Then we'll get a central extension of $L\mathfrak{gl}_n$.

<u>Claim</u>. This is the extension defining the affine Lie algebra. Namely,

$$\alpha(a(t), b(t)) = \operatorname{Res}_{t=0} (a'(t), b(t))\,dt$$

$$= \sum_k k\,\mathrm{tr}(a_k b_{-k}).$$

<u>Proof</u>. Easy computation.

So we get an inclusion of $\widehat{\mathfrak{gl}}_n = L\mathfrak{gl}_n \oplus \mathbb{C}K$ into $\mathfrak{a}_\infty$, and in particular the spaces $\mathcal{F}^{(m)} \cong \mathcal{B}^{(m)}$ become representations of $\widehat{\mathfrak{gl}}_n$ at level 1, i.e with $K=1$.

Also consider $L\mathfrak{sl}_n = \mathfrak{sl}_n[t,t^{-1}] \hookrightarrow \overline{\mathfrak{a}}_\infty$, and $\widehat{\mathfrak{sl}}_n = L\mathfrak{sl}_n \oplus \mathbb{C}k$. This is also represented in $\mathcal{F}^{(m)}, \mathcal{B}^{(m)}$ with level 1.

Now consider the derivation $d : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ given by $d(a(t)) = t a'(t)$, $d(K) = 0$.

We can consider a semidirect product $\widetilde{\mathfrak{g}} = \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$.

<u>Prop.</u> The representation $F^{(m)}$ of $\widehat{sl}_n, \widehat{gl}_n$ extends uniquely to $\tilde{sl}_n, \tilde{gl}_n$ by setting $d\varphi_m = 0$.

<u>Proof</u>. Set $dw = \text{degree}(w)\, w$ for any wedge w, where degree(w) is defined appropriately. ('principal gradation')

Recall now that $\widehat{sl}_n, \widetilde{gl}_n$ are graded Lie algebras. E.g.

$\widehat{sl}_n = n_+ \oplus \mathfrak{h} \oplus n_-$, where $\tilde{\mathfrak{h}} = \mathbb{C}d \oplus \mathfrak{h} \oplus \mathbb{C}K$

$\mathfrak{h}$ = diagonal matrices of trace 0.

We have a basis $h_i = E_{ii} - E_{i+1,i+1}$ of $\mathfrak{h}$
$n-1 \geq i \geq 1$

So h_i, $h_0 = K - (h_1 + \cdots + h_{n-1})$, d is a basis of $\tilde{\mathfrak{h}}$. Define the elements

$\tilde{\omega}_m \in \tilde{\mathfrak{h}}^*$, $i = 0, \ldots, n-1$ by

$\tilde{\omega}_m(h_j) = \delta_{mj}$, $\tilde{\omega}_m(d) = 0$.

Similarly, for $\widetilde{gl}_n$ we have weights $\tilde{\omega}_m$ defined by

$$\tilde{\omega}_m(E_{jj}) = \begin{cases} 1, & j \leq m \\ 0, & j > m \end{cases} + \left[\frac{m}{n}\right]$$

<u>Proposition</u> $B^{(m)} = F^{(m)}$ is an irreducible representation of $\widetilde{gl}_n$ with highest

weight ω_m, where m is taken modulo n.

Proof 1) Let us show that $\mathcal{B}^{(m)}$ is irreducible. Consider the elements T^i. They are contained in $L\mathfrak{gl}_n$, namely

$$T = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ t & & & 1 \end{pmatrix}$$ and T^i is the i-th power of T. But they generate $\mathcal{A}$ (Heisenberg algebra) for which $\mathcal{B}^{(m)}$ is irreducible.

2) Highest weight:

$$\rho(E_{ii})\psi_m = \begin{cases} \psi_m, & i \leq m \\ 0, & i > m, \end{cases}$$

so $$\hat{\rho}\Big(\sum_{i \equiv j \bmod n} E_{ii}\Big)\psi_m = \left(\left[\frac{m}{n}\right] + \begin{cases} 1, & j \leq m \\ 0, & j > m \end{cases}\right)\psi_m \quad j = 1, \dots, n$$

As a representation of $\widehat{sl}_n$, $F^{(m)} = \mathcal{B}^{(m)}$ is not an irreducible repr. Indeed, T^{ni} commute with the action of $\widehat{sl}_n$, but aren't scalars. To deal with this, define $\mathcal{B}_0^{(m)}$ to be $\mathbb{C}[x_i, i \geq 1, i \not\equiv 0 \bmod n]$

$\mathcal{B}_0^{(m)}$ is a subrepresentation which is irreducible for $\widetilde{sl}_n$ with highest weight $\widetilde{\omega}_m$. generated by T^{ni}

<u>Proof.</u> $\widehat{gl}_n = \widehat{sl}_n \oplus \mathcal{A}_n / (K_1 = K_2)$

$$\mathcal{B}^{(m)} = \mathcal{B}_0^{(m)} \otimes F^{(n)}$$

$\mathcal{B}_0^{(m)} = \langle x_i,\ i \not\equiv 0 \bmod n \rangle$, $F^{(n)}$ Fock space $= \langle x_{nj} \rangle$.

and $\mathcal{B}_{(0)}^{(m)}$ is stable under $\widetilde{sl}_n$.

Using this, we want to obtain a classification of unitary highest weight representations of $\widetilde{sl}_n$.

<u>Theorem.</u> L_λ for $\widehat{sl}_n$ is unitary

$$\Longleftrightarrow \lambda = k_0 \widetilde{\omega}_0 + \cdots + k_{n-1} \widetilde{\omega}_n, \quad k_i \in \mathbb{Z}_{\geq 0}.$$

<u>Proof.</u> First, if $k_i \in \mathbb{Z}_+$ then L_λ is a composition factor in

$$L_{\widetilde{\omega}_0}^{\otimes k_0} \otimes \cdots \otimes L_{\widetilde{\omega}_{n-1}}^{\otimes k_{n-1}},$$

and $L_{\widetilde{\omega}_i}$ are unitary, so L_λ is unitary. To prove the converse, we use the following lemma.

Let L_m be an irreducible highest wt repr of sl_2, which is unitary with respect to the involution $e^+ = f$, $h^+ = h$. Then $m \in \mathbb{Z}_+$.

Pf. Indeed, $(f^r v_m, f^r v_m) =$

$$= \prod_{i=1}^{r} (m-i+1) i.$$

So for unitarity we need $m-i+1 \geq 1$ for $1 \leq i \leq r$, whenever $f^r v_m \neq 0$, which means we can only have finite dim. repr. which are indeed unitary.

Now restrict L_λ to the sl_2-subalgebra generated by $e_j = \sum_{i \equiv j \bmod n} E_{i\,i+1}$ and $f_j = \sum_{i \equiv j \bmod n} E_{i+1\,i}$. The integrality condition of the lemma then tells us that $\lambda(h_j) \in \mathbb{Z}_{\geq 0}$, which implies the statement.

(in LSl_n, $e_j = E_{j\,j+1}$ for $j \geq 1$ and $e_0 = E_{n1} t$ similarly $f_0 = E_{n1} t^{-1}$ $f_j = E_{j+1\,j}$)