Recall that the Verma module $M_{\lambda}$, $\lambda = (c, h)$ over $V$ has a contravariant form $(\cdot, \cdot)$, also called shapevalen form. It has the contravariance property $(L_n v, w) = (v, L_{-n} w)$ and is symmetric.

Also, different degrees are orthogonal with respect to this form. Let us consider the determinant polynomial of this form in degree $n$, $\det_n (c, h)$. As we remember, it is uniquely determined up to scaling.

Prop. (proved before) $\det_n (c, h) = 0$ \(\Rightarrow\) there is a $\neq 0$ singular vector of degree $\leq n$. Thus, $M_{\lambda}$ is irreducible.

$\Rightarrow$ $\det_n (c, h) \neq 0$ $\forall n$

Corollary $\det_m (c, h) = 0 \Rightarrow \det_{m+1} (c, h) = 0$

so if $X_m = \sqrt{\det_m}$ then $X_m \leq X_{m+1}$.

(in fact, we'll see that $\det_{m+1}$ is divisible but $\det_m$)

Ex. $\det_1 = 2h$, $\det_2 = 2h (16h^2 + 2hc - 10h + c)$.
Prop. $M(c, h)$ is unitary

$\Rightarrow \det_m(c, h) > 0 \ \forall m$
(if we use the Hermitian form)

Theorem. Fix $c$. Then

$$\det_m(c, h) = K_m h \sum_{1 \leq r, s \leq m} p(m-rs) + \text{lower terms}$$

where $K_m \neq 0$.

Proof. This follows from a general theorem before (we study asymptotics for fixed $c$ and $h \to \infty$, which corresponds to the "Heisenberg limit" with $c = 0$ and $h$ finite). So we just need to compute the degree with respect to $h$. Suppose we have a partition $\lambda$ of $m$:

$$m = k_1 + 2k_2 + \cdots + pk_p, \ k_i = k_i(\lambda) \ \text{multiplicity of i}$$

Then we have a monomial $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ which contributes $k_1 + 2k_2 + \cdots + pk_p$ to the degree. So the degree is

$$D_m = \sum_{\lambda} \sum_{i} k_i(\lambda).$$

Claim. This can be written as

$$\sum_{r, s: 1 \leq r, s \leq m} s m(r, s)$$
Indeed, we have \( m(r,s) \) partitions with \( r \) occurring \( s \) times, so total number of times \( r \) occurs is \( \sum_{1 \leq s \leq m} sm(r,s) \).

**Lemma.** \[ \sum_{1 \leq s \leq m} sm(r,s) = \sum_{r} p(m-rs) \]

**Proof.** \[ m(r,s) = p(m-rs) - p(m-r(s+1)), \] so the result follows.

The theorem is proved.

Now we will discuss zeros of the determinant polynomials.

**Thm.** (Kac; another proof given by Fergin and Fuchs)

Let, for \( 1 \leq r, s \leq m, rs \leq m \)

\[ h_{r,s}(c) = \frac{1}{48} \left( (13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)(r^2-s^2)^2 - 24rs - 2c} \right) \]

Then

\[ \det_m(c,h) = K_m \prod_{1 \leq r, s \leq m} (h - h_{r,s}(c)) p(m-rs) \]

**Remark.** For each \( r,s \), we have to choose a branch of square root, but the other branch corresponds to switching \( r \) and \( s \).

For the proof, we need the following lemma.
Lemma. Let $A(t)$ be a matrix whose entries are polynomial in $t$ such that $\dim \ker A(0) \geq N$. Then $\det A(t)$ is divisible by $t^N$.

Proof. Let $v_1, \ldots, v_N$ be linearly independent vectors in $\ker A(0)$. Complete them to a basis. Then $A$ written in this basis has the first $N$ columns divisible by $t$, which implies the statement.

Now we go back to the proof.

Thm. (to be proved later). One has $\det_{rs} h, s(c), c = 0$ (in fact, there is a singular vector of degree $rs$ for $h = h, s(c)$).

Now, using this theorem, we conclude that $\det_{rs} c, h$ is divisible by $\text{singular vector } (h-h, s(c))$ as we have a vector in degree $rs$, and it generates a Verma submodule, which in degree has dimension $\geq p(m-rs)$. Since the total degree and the leading term of $\det_{rs}(h,s)$ is known, and it equals $2p(m-rs)$.
and $K_m$, we are done.

The formula can be rewritten as:

$$
\det_\mathcal{M}(c, h) = K_m \prod_{1 \leq r < s \leq m} (h - h_{r,s}(c)) (h - h_{s,r}(c))
$$

where $h_{r,s}(c)$ is linear in $c$ (defines lines) and

$(h - h_{r,s}(c)) (h - h_{s,r}(c))$ is quadratic (defines hyperbolas $\Gamma_{r,s}$)

So outside of a countable set of lines and hyperbolas, $\mathcal{M}_{c,h}$ is irreducible.

**Corollary** 1) Let $h > 0, c > 1$. Then $L_{c,h}$ is unitary.

2) Let $h > 0, c > 0$, then $L_{c,h} \approx M_{c,h}$.

**Proof** It's easy to see that the hyperbolas and lines don't intersect this open region (the functions are positive there):

$$
\left(h - \frac{(r-s)^2}{4}\right)^2 + \frac{h^{c-1}}{24} (r^2 + s^2 - 2) + \frac{1}{576} (r^2 - 1)(s^2 - 1)(c-1)^2
$$

$$
+ \frac{1}{98} (c-1)(r-s)^2 (25+1) = 0 \quad \text{(hyperbolas)}
$$

$$
h + \frac{(r^2 - 1)(c-1)}{24} = 0 \quad \text{(lines)}
$$
This means that the second statement holds. Since we already know that \( L_{c,F} \) is unitary for some points of this open region, it should also be true for all points by deformation argument, and also on the boundary by continuity (we use that a positive form cannot become nonpositive without passing through degenerate forms, and limit of a positive form is a nonnegative form).

Now we know a lot about unitary representations. One can show all of them should occur above the lines and in between the branches of the hyperboloid (this is where the region \( h > 0, c > 1 \) is). (If we cross, determinant changes sign).

By detailed analysis of these regions, Friedan-Quigley-Sheet-ker showed that the only remaining possible points
are the following:

\[ c(m) = 1 - \frac{6}{(m+2)(m+3)} \quad , \quad m \geq 0 \]

\[ h_{j,r}(m) = \frac{((m+3)r - (m+2)s)^2}{4(m+2)(m+3)} \quad , \quad 1 \leq r \leq m+1 \]

We will show that for these points the line has indeed unitary reps. They are called "discrete series". We will not prove that these are the only points, but let us derive the following corollaries.

Prop. \( c = 0, \; L_c, h \) unitary \( \Rightarrow h = 0 \)

pf. Consider the form on the 2-d space \( \langle L_{-N^2}, L_{-2N^2} \rangle \). The determinant is \( 4N^2h^2(8h-5N) \), so for this to be \( \geq 0 \) we need \( h = 0 \).

Prop. \( L_0, h = M_0, h \Leftrightarrow h \neq \frac{m^2-1}{24} \quad , \quad m \in \mathbb{Z} \)

Prop. \( L_1, h = M_1, h \Leftrightarrow h \neq \frac{m^2}{24} \quad , \quad m \in \mathbb{Z} \)

pf. Direct computation with the determinant formula.
Consider the loop algebra $\mathfrak{gl}_n = \mathfrak{gl}_n[\mathbb{C}[t, t^{-1}]]$. It acts on $\mathbb{C}[t, t^{-1}]$. This representation has basis $e_{it^k}$ where $e_0, \ldots, e_{n-1}$ is a basis of $\mathbb{C}^n$. Denote $e_{it^k}$ by $v_i - k \nu$. Then we get an identification of $\mathbb{C}^n[\mathbb{C}[t, t^{-1}]]$ with $V = \mathbb{C}^\infty$, the tautological representation of $\mathfrak{gl}_n$. Now consider $\alpha(t) = \sum a_k t^k$ (finite sums and we want to express it as an element of $\mathfrak{gl}_n$ (acting on $\mathbb{C}^\infty = \mathbb{C}^n[\mathbb{C}[t, t^{-1}]]$). When we do this, we get a block matrix

$$
\begin{pmatrix}
 a_0 & a_1 & a_2 & a_3 & \\
 a_{-1} & a_0 & a_1 & a_2 & \\
 a_{-2} & a_{-1} & a_0 & a_1 & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 \end{pmatrix}
$$

Indeed, if we apply this to a vector $(0, 0, 0, \ldots, 0)$, we get

$$
\begin{pmatrix}
 a_0 & a_1 & a_2 & a_3 & \\
 a_{-1} & a_0 & a_1 & a_2 & \\
 a_{-2} & a_{-1} & a_0 & a_1 & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 \end{pmatrix}
\begin{pmatrix}
 0 \\
 0 \\
 0 \\
 v \\
 v \\
 \end{pmatrix}
= \sum a_k t^k \nu
$$

Thus we get an embedding $\mathfrak{gl}_n \rightarrow \mathfrak{gl}_\infty$. 

Now recall that \( \overline{\alpha}_{\infty} \) carries a central extension defined by the 2-cocycle \( \alpha \) of \( \log \mathfrak{g}_n \). Then we'll get a central extension of \( \log \mathfrak{g}_n \).

**Claim.** This is the extension defining the affine Lie algebra. Namely,

\[
\alpha(a(t), b(t)) = \lim_{t \to 0} (a'(t), b(t)) \, dt
\]

\[= \sum_k k \text{tr}(a_k b_{-k}).\]

**Proof.** Easy computation.

So we get an inclusion of \( \hat{\mathfrak{g}}_n = \log \mathfrak{g}_n \oplus \mathfrak{c}_k \) into \( \mathfrak{a}_\infty \), and in particular the spaces \( F^{(m)} \cong B^{(m)} \) become representations of \( \hat{\mathfrak{g}}_n \) at level 1, i.e., with \( k = 1 \).

Also consider \( L \mathfrak{sl}_n = \mathfrak{sl}_n \oplus [t, t^{-1}] \leq \widehat{\mathfrak{g}}_\infty \), and \( \hat{\mathfrak{g}}_n = L \mathfrak{sl}_n \oplus \mathfrak{c}_k \). This is also represented in \( F^{(m)}, B^{(m)} \) with level 1.

Now consider the derivation \( d : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}} \) given by \( d(a(t)) = t a'(t) \) \( d\mathbf{R} = 0 \).

We can consider a semidirect product \( \hat{\mathfrak{g}} = \mathfrak{c}_k \times \hat{\mathfrak{g}} \).
Prop. The representation $\tilde{\rho}^{(m)}$ of $\mathfrak{gl}_n$ extends uniquely to $\tilde{\varphi}_{\mathfrak{g}_n}$ by setting $d_{\varphi} = 0$.

Proof. Set $d_\varphi = \text{degree}(w) \cdot w$ for any weight $w$, where degree$(w)$ is defined appropriately.

Recall now that $\tilde{\mathfrak{h}}_n$, $\tilde{\mathfrak{g}}_n$ are graded Lie algebras. E.g.,

$\tilde{\mathfrak{h}}_n = \mathfrak{h}_+ \oplus \mathfrak{h} \oplus \mathfrak{h}_-$, where $\mathfrak{h} = \mathfrak{c} \mathfrak{d} \oplus \mathfrak{g} \oplus \mathfrak{c} \mathfrak{k}$, $\mathfrak{g} = \text{diagonal matrices of trace 0}$.

We have a basis $h_i = E_{ii} - E_{i+i+1}$ of $\mathfrak{h}$.

So $h_i$, $h_0 = K - (h_1 + \ldots + h_{n-1})$, $d$ is a basis of $\mathfrak{g}_\ast$. Define the elements

$\tilde{\omega}_m \in \mathfrak{g}_\ast$, $i = 0, \ldots, n-1$, by

$\tilde{\omega}_m(h_i) = \delta_{ij}$, $\tilde{\omega}_m(d) = 0$.

Similarly, for $\mathfrak{gl}_n$ we have weights $\tilde{\omega}_m$ defined by

$\tilde{\omega}_m(E_{ij}) = \begin{cases} \frac{1}{i}, & k \leq m + \left[ \frac{m}{n} \right] \\ 0, & j > m \end{cases}$

Proposition $\tilde{\varphi}^{(m)} \circ \rho^{(m)}$ is an irreducible representation of $\mathfrak{gl}_n$ with highest
weight \( w_m \), where \( m \) is taken modulo \( n \).

**Proof.**

1) Let us show that \( B^{(m)} \) is irreducible. Consider the elements \( T_i \) of \( G \). They are contained in \( L \) of \( L \), namely \( T = \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix} \) and \( T_i \) is the \( i \)-th power of \( T \). But they generate \( A \) (Heisenberg algebra) for which \( B^{(n)} \) is irreducible.

2) Highest weight:

\[
\rho(E^{ii}) \psi_m = \begin{cases} \psi_m, & i \leq m \\ 0, & i > m, \end{cases}
\]

so

\[
\rho \left( \sum_{i \equiv j \mod n} E^{ii} \right) \psi_m = \left( \left\lfloor \frac{m}{n} \right\rfloor + \left\{ 0, \frac{i}{j} \leq m \right\} \right) \psi_m, \quad j = 1, \ldots, n
\]

As a representation of \( G \), \( F^{(m)} \) is not an irreducible repr. Indeed, \( T_i \) commute with the action of \( G \), but aren't scalars.

To deal with this, define \( B^{(m)}_0 \) to be \( \mathbb{C}[x_i, i \geq 1, i \neq 0 \mod m] \).
$B_{(m)}$ is a subrepresentation which is irreducible for $\hat{gl}_n$ with highest weight $\tilde{\omega}_m$. It is generated by $T^{i_1} T^{i_2} \cdots T^{i_n}$.

Proof. \[ \hat{gl}_n = \mathfrak{sl}_n \oplus \mathfrak{h} \quad (K_1 = K_2) \]

$B_{(m)} = B_{(m)} \otimes F(n)$

\[ \langle x^i, i \neq 0 \mod n \rangle \langle n^j \rangle. \]

and $B_{(m)}$ is stable under $\hat{gl}_n$.

Using this, we want to obtain a classification of unitary highest weight representations of $\hat{gl}_n$.

Theorem. \( L_\lambda \) for $\hat{gl}_n$ is unitary

\((\Rightarrow)\) \quad $\lambda = k_0 \tilde{\omega}_0 + \cdots + k_{n-1} \tilde{\omega}_{n-1}$, \quad $k_i \in \mathbb{Z}_{\geq 0}$.

Proof. First, if $k_i \in \mathbb{Z}_+$ then $L_\lambda$ is a composition factor in

$\bigotimes_{k_0}^{k_{n-1}} L(\tilde{\omega}_0) \otimes \cdots \otimes L(\tilde{\omega}_{n-1})$, and $L(\tilde{\omega}_i)$ are unitary, so $L_\lambda$ is unitary. To prove the converse, we use the following lemma.
Let $L_m$ be an irreducible highest weight rep of $\mathfrak{sl}_2$, which is unitary with respect to the involution $e^+ = f$, $h^+ = h$. Then $m \in \mathbb{Z}_+$. 

Proof. Indeed, \( (f^* v_m, f^* v_m) = \frac{1}{\prod_{i=1}^{r} (m-i+1)i} \).

So for unitarity we need $m-i+1 \geq 1$ for $1 \leq i \leq r$, whenever $f^* v_m \neq 0$, which means we can only have finite dim. repr., which are indeed unitary.

Now restrict $L_\lambda$ to the $\mathfrak{sl}_2$-subalgebra generated by $e_j^+ = \sum_{i=j+1 \text{ mod } n} E_{ii+1}$ and $f_j^- = \sum_{i=j \text{ mod } n} E_{ii+1}$. The integrality condition of the lemma then tells us that 

\[ \lambda(h_j) \in \mathbb{Z} \geq 0, \]

which implies the statement.