# Lecture 11

Recall: $X(u) = \sum_i \xi_i u^i$, $X^*(u) = \sum_i \xi_i^* u^{-i}$

The equation $S(\tau \otimes \tau) = 0$ can be replaced by $CT(X(u)\tau \otimes X^*(u)\tau) = 0$, where $CT$ is the constant term.

Now we go from $F^{(0)}$ to $B^{(0)}$ by $\sigma^{-1}$.

We know that $X(u)$ goes to $\Gamma(u)$ and $X^*(u)$ to $\Gamma^*(u)$, so we get

$$CT(\Gamma(u)\tau \otimes \Gamma^*(u)\tau) = 0,$$

where now $\tau \in \mathbb{C}[x_1, x_2, x_3, \dots] = F$

We realize $F \otimes F$ as $\mathbb{C}[x_1', x_1'', x_2', x_2'', \dots]$

Then the Plücker equation becomes

$$CT\left(u e^{\sum_{j\geq 1} u^j x_j'} e^{-\sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j'}} e^{-\sum_{j\geq 1} u^j x_j''} e^{\sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j''}} \right.$$

$$\left. \tau(x_1', x_2', \dots)\tau(x_1'', x_2'', \dots)\right) = 0,$$

or

$$CT\left(u e^{\sum u^j (x_j' - x_j'')} e^{\sum \frac{u^{-j}}{j}\left(\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''}\right)} \tau(x')\tau(x'')\right) = 0$$

Change of variables: $\begin{cases} x' = x - y \\ x'' = x + y \end{cases} \iff \begin{cases} x' - x'' = -2y \\ \frac{\partial}{\partial x'} - \frac{\partial}{\partial x''} = -\frac{\partial}{\partial y} \end{cases}$

$$CT\left(u e^{-2\sum u^j y_j} e^{\sum \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}}\right) \tau(x-y)\,\tau(x+y)) = 0$$

For three polynomials $P, f, g$ of infinitely many variables, denote by $A(P,f,g)$ the function

$$A(P,f,g)(x) = P\left(\frac{\partial}{\partial z}\right)\left(f(x-z)\,g(x+z)\right)\Big|_{z=0}.$$

$x = x_1, x_2, \ldots$

$z = z_1, z_2, \ldots$

$\frac{\partial}{\partial z} = \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots$

<u>Ex.</u> $P(w) = w_1 \Rightarrow$

$A(P,f,g) = -\frac{\partial f}{\partial x_1} g + f \frac{\partial g}{\partial x_1}$

<u>Lemma.</u> If $P$ is odd then $A(P,f,f) = 0$.

<u>Pf.</u> replacing $z$ by $-z$, we get $A = -A$.

<u>Remark.</u> In general $A(P,f,g) = A(\underline{P}, g, f)$, where $\underline{P}(w) = P(-w)$.

<u>Theorem.</u> (Hirota bilinear relations)

$\tau \in \Omega$ if and only if

$\overset{\#}{0}$

$$A\Big(\sum_{j=0}^{\infty} S_j(-2y)\, \overset{\boxed{G(x,y)}}{\overset{\shortparallel}{S_{j+1}}}(\tilde{x})\, e^{\sum_{s\geq 1} y_s x_s}, \tau, \tau\Big) = 0$$

(where $y$ are parameters), with

$\tilde{x}_1 = x_1,\ \tilde{x}_2 = \frac{x_2}{2},\ \tilde{x}_3 = \frac{x_3}{3}, \ldots$

<u>Proof.</u> $CT\left(u\, e^{-\sum 2u^j y_j}\, e^{\sum \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}}\, \tau(x+y)\,\tau(x-y)\right)$

$= CT\left(u\, e^{-\Sigma(\text{-''-})}\, e^{\Sigma(\text{-''-})}\, \tau(x+y+t)\,\tau(x-y-t)\right)\Big|_{t=0}$

Now let $\tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \ldots\right)$

we have $e^{-\sum 2u^j y_j} = \sum_{k\geq 0} u^k S_k(-2y)$

$$e^{\sum \frac{u^{-j}}{j}\frac{\partial}{\partial t_j}} = \sum_{\ell \geq 0} u^{-\ell} S_\ell(\tilde{\partial}_t)$$

so $CT(-\text{''}-) =$

$$= CT\left(u \sum_{k\geq 0} u^k S_k(-2y) \sum_{\ell\geq 0} u^{-\ell} S_\ell(\tilde{\partial}_t)\, \tau(x+y+t)\tau(x-y-t)\right)\Big|_{t=0}$$

$$= \left(\sum_{k\geq 0} S_k(-2y) S_{k+1}(\tilde{\partial}_t) e^{\sum y_s \frac{\partial}{\partial t_s}} \tau(x+t)\tau(x-t)\right)\Big|_{t=0}$$

$$= A\left(\sum_{k\geq 0} S_k(-2y) S_{k+1}(\tilde{x}) e^{\sum y_s x_s}, \tau, \tau\right)(x). \blacksquare$$

This relation is actually a bunch of relations corresponding to particular monomials in $y$ (called the Hirota bilinear relations).

1) degree zero in $y$ : $k=0$

$A(\underset{\substack{\| \\ x_1}}{S_1(\tilde{x})}, \tau, \tau) = 0$ — always true since $x_1$ is odd

2) degree 1 in $y$ (usual notion of degree, so $y_r$).

The coefficient of $y_r$ in $G(x,y)$ is

$$x_1 x_r - 2S_{r+1}(\tilde{x}).$$

Indeed, either $y_r$ comes from the exponential, then $k=0$, $s=r$, so we get $x_1 x_r$, or it comes from $S_k(-2y)$, then $k=r$, and we get $-2S_{r+1}(\tilde{x})$, as $S_k(z) = z_k + \dots$

So we have the differential equations

$$A(x_1 x_r - 2S_{r+1}(\tilde{x}), \tau, \tau) = 0,$$

which go under the name of the Kadomtsev-Petviashvili hierarchy.

Let's study these equations in more detail. Let $T_r(x) = x_1 x_r - 2S_{r+1}(\tilde{x})$

$$T_1(x) = x_2$$

$$T_2(x) = -\frac{x_1^3}{3} - \frac{2x_3}{3}$$

$$T_3(x) = \frac{x_1 x_3}{3} - \frac{x_4}{2} - \frac{x_2^2}{4} - \frac{x_1^4}{12} - \frac{x_1^2 x_2}{2}$$

Recall that odd polynomials give trivial equations, so the first nontrivial equation comes from $T_3$:

$$A(T_3, \tau, \tau) = 0.$$

and in fact it is

$$A\left(\frac{x_1x_3}{3}-\frac{x_2^2}{4}-\frac{x_1^4}{12}, \tau, \tau\right)=0. \qquad (*)$$

So

$$\left(\frac{\partial^4}{\partial w_1^4}+3\left(\frac{\partial}{\partial w_2}\right)^2-4\frac{\partial^2}{\partial w_1 \partial w_3}\right)\tau(\vec{z}-\vec{w})\tau(\vec{z}+\vec{w})\Big|_{w=0}=0$$

Note that the equation involves only three first variables. So set

$x_1=x$, $x_2=y$, $x_3=t$, $x_m=c_m$, $m \geqslant 4$

<u>Prop.</u> Let $U = 2\frac{\partial^2}{\partial x^2}\log\tau$.
Then the above equation $(*)$ for $\tau$ is equivalent to the KP equation for $U$:

$$\frac{3}{4}\frac{\partial^2 U}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t}-\frac{3}{2}U\frac{\partial U}{\partial x}-\frac{1}{4}\frac{\partial^3 U}{\partial x^3}\right)$$

<u>Pf.</u> homework.

<u>Corollary.</u> If $S_\lambda$ is the Schur polynomial corresponding to a partition $\lambda$, then $2\frac{\partial^2}{\partial x^2}\log S_\lambda(x, y, t, c_4, c_5, c_6, \ldots)$ satisfies the KP equation (and in fact the whole KP hierarchy)

Pf. This holds because we know that monomials in $F^{(0)}$ are in $\Omega$, and they correspond to $S_\lambda \in B^{(0)}$.

Now let us construct other solutions of the KP equations.

Recall $= u:\Gamma(u)\Gamma^*(v):$

$$\Gamma(u,v) = e^{\sum \frac{u^j - v^j}{j} a_{-j}} e^{-\sum \frac{u^{-j} - v^{-j}}{j} a_j}$$

defining action of $gl_\infty$ on $B$ (up to a scalar function).

Till now, we considered this as a power series, but now we'll consider it as a function in $u$ and $v$.

Proposition. $\tau \in \Omega \Rightarrow (1 + a\Gamma(u,v))\tau \in \Omega$.

Corollary

$(1 + a_1\Gamma(u_1, v_1)) \cdots (1 + a_n \Gamma(u_n, v_n)) \mathbb{1} \in \Omega$.

To prove the proposition, introduce the notation $\Gamma_+ = \Gamma$, $\Gamma_- = \Gamma_*$.

We'd like to calculate the

normal ordered product

$$:\Gamma_{\varepsilon_1}(u_1)\cdots\Gamma_{\varepsilon_k}(u_k):$$

relating it to the usual product

<u>Prop.</u>

$$\Gamma(u)\Gamma(v) = (u-v):\Gamma(u)\Gamma(v):$$

$$\Gamma(u)\Gamma^*(v) = \frac{1}{u-v}:\Gamma(u)\Gamma^*(v):$$

$$\Gamma^*(u)\Gamma(v) = \frac{1}{u-v}:\Gamma^*(u)\Gamma(v):$$

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where $\frac{1}{u-v} = \frac{1}{u}\cdot\frac{1}{1-\frac{v}{u}} = \frac{1}{u}+\frac{v}{u^2}+\frac{v^2}{u^3}+\cdots$

<u>Pf.</u> Straightforward (similar to computation $\Gamma(u)\Gamma^*(v)$ before)

<u>Corollary.</u>

$$\Gamma_{\varepsilon_1}(u_1)\cdots\Gamma_{\varepsilon_k}(u_k) = \prod_{i<j}(u_i-u_j)^{\varepsilon_i\varepsilon_j}:\Gamma_{\varepsilon_1}(u_1)\cdots\Gamma_{\varepsilon_k}(u_k):$$

<u>Pf.</u> same

<u>Cor.</u> All matrix elements of $\Gamma_{\varepsilon_1}(u_1)\cdots\Gamma_{\varepsilon_k}(u_k)$ are rational functions, and series converge for $|u_1|>|u_2|>\cdots$ They have the form $P(u)\prod_{i<j}(u_i-u_j)^{\varepsilon_i\varepsilon_j}$, where $P$ is a Laurent polynomial.

Pf. This follows from the fact that $\langle w_1 : \Gamma \cdots \Gamma : w_2 \rangle$ is always a Laurent polynomial

Cor.

$$\Gamma(u',v')\Gamma(u,v) = \frac{(u'-u)(v'-v)}{(v'-u)(u'-v)} :\Gamma(u',v')\Gamma(u,v):$$

Cor.

$\langle w_1, \Gamma(u',v')\Gamma(u,v) w_2 \rangle$ is a series that converges to a rational function which is regular when $u'=u$, $v'=v$, and if $u \neq v$, $u, v, u', v' \neq 0$,

$$\lim_{\substack{u' \to u \\ v' \to v}} \Gamma(u',v')\Gamma(u,v) = 0$$

(which is to say that this holds after taking matrix elements).

We can rewrite this more informally as $\Gamma(u,v)^2 = 0$.

Now we can prove the theorem.

Thm: Let $\tau \neq 0$, $\tau \in \Omega$. Then $(1 + a\Gamma(u,v))\tau \in \Omega_{u,v}$, $(a \in \mathbb{C})$ where $\Omega_{u,v} = \{\tau \in \mathcal{B}^0((u,v))$ s.t. $S(\tau \otimes \tau) = 0\}$

Proof idea: $(1+a\Gamma(u,v))\tau = e^{a\Gamma(u,v)}\tau$, and $e^{a\Gamma(u,v)} \in GL(\sigma)$ since $\Gamma(u,v)^2 = 0$.

But this is not quite rigorous.

Rigorous proof:

$$S\left[(1+a\Gamma(u,v))\tau \otimes (1+a\Gamma(u,v))\tau\right] = S(\tau\otimes\tau)$$

$$= aS\left(\Gamma(u,v)\tau\otimes\tau + \tau\otimes\Gamma(u,v)\tau\right)$$

$$+ a^2 S\left(\Gamma(u,v)\tau\otimes\Gamma(u,v)\tau\right).$$

The first summand is zero.

The second summand is zero since $S$ is invariant under $gl_\infty$.

So it remains to show that the last summand is zero. But the last summand is (up to $a^2$ factor)

$$\lim_{\substack{u'\to u\\ v'\to v}} \frac{1}{2}\left(S\left(\Gamma(u,v)\tau\otimes\Gamma(u',v')\tau\right) + S\left(\Gamma(u',v')\tau\otimes\Gamma(u,v)\tau\right)\right)$$

$$= \lim_{\substack{u'\to u\\ v'\to v}} \frac{1}{2}\left(S\left(\Gamma(u',v')\otimes 1 + 1\otimes\Gamma(u',v')\right)\left(\Gamma(u,v)\otimes 1 + 1\otimes\Gamma(u,v)\right)\tau\otimes\tau\right.$$

$$- \lim_{\substack{v'\to v\\ u'\to u}} \frac{1}{2} S\left(\Gamma(u',v')\Gamma(u,v)\tau\otimes\tau + \tau\otimes\Gamma(u',v')\Gamma(u,v)\tau\right)$$

and both summands are zero.

<u>Corollary</u> If

$$(1+a_1\Gamma(u_1,v_1))\cdots(1+a_n\Gamma(u_n,v_n))\,\mathbb{I} = \tau$$

then $2\frac{\partial^2}{\partial x_1^2}\log\tau$ is a convergent series and is a solution of the KP hierarchy

Ex.

$$(1+a\Gamma(u,v))\mathbb{I} = 1+a e^{(u-v)x+(u^2-v^2)y+(u^3-v^3)t+C}$$

So after calculation

$$2\frac{\partial^2}{\partial x^2}\log\tau = \frac{(u-v)^2}{2}\,\frac{1}{\cosh^2\frac{1}{2}\left((u-v)x+(u^2-v^2)y+(u^3-v^3)t+C\right)}$$

$= U(x,y,t)$

This is the 1-soliton solution of the K hierarchy (and in particular KP equation).

So if $u=-v$, we get a solution independent of $y$:

$$U(x,t) = \frac{2u^2}{\cosh(2ux+u^3t)}$$

which is the usual soliton wave. It satisfies $\frac{\partial}{\partial x}(KDV)=0$, so KDV. (as decays at $\infty$)