Now I want to discuss the relationship of all this with integrable systems. One of the most important integrable PDEs is the KdV equation

$$u_t = \frac{3}{2} uu_x - \frac{1}{4} u_{xxx} \quad (\text{cosh}^2 \frac{3}{2}, \frac{1}{4} \text{ can be normalized to } 1)$$

This was studied in 1895 by Korteweg and de Vries as an equation describing notion of waves in shallow water. One of the solutions is the traveling wave solution:

$$f(t) = \frac{2u^2}{\cosh^2 (x + vt)}$$

Such solution is called a Soliton (solitary wave). It was first observed by J.S. Russell in 1834. Generalization: Kadomtsev-Petviashvili equation

$$u_{yy} = (u_t - \frac{3}{2} uu_x - \frac{1}{4} u_{xxx})_x$$

This describes 2-dimensional waves.
We'll construct many solutions of these equations using rep. theory of \( \infty \)-dimensional Lie algebras. For this purpose we'll need infinite Grassmannians, so let us first recall about finite dimensional Grassmannians.

Let \( V \) be a finite dimensional vector space, with basis \( v_1, \ldots, v_n \). Then \( GL(V) \) acts on \( \Lambda^k V \), and the highest weight vector is \( v_1 \wedge \ldots \wedge v_k \). Denote by \( \mathcal{S}_k \) the orbit of action of \( GL(V) \) on \( v_1 \wedge \ldots \wedge v_k \). So \( \mathcal{S}_k \) is the set of decomposable wedges:

\[
\mathcal{S}_k = \{ x \in \Lambda^k V, \exists x_1, \ldots, x_k \in V, x = x_1 \wedge \ldots \wedge x_k \}.
\]

Prop. \( \mathcal{S}_k \) is clear. (Note that \( x \neq 0 \) implies that \( x_1, \ldots, x_k \) are linearly independent.)

Def. The Grassmannian \( Gr(k, V) \) is the set of all \( k \)-dim. subspaces of \( V \).

Plücker embedding: \( Pl: Gr(k, V) \to \mathbb{P} \Lambda^k V \)

\( E \in V, x_1, \ldots, x_k \) basis \( \Rightarrow Pl(E) = x_1 \wedge \ldots \wedge x_k \).
Exer. Show that \( PE \) is injective.

Clearly, \( \text{Im}(PE) = \mathcal{S}/\mathcal{E}^* \)

So \( \mathcal{S}/\mathcal{E}^* \cong \text{Gr}(k, V) \).

\( \mathcal{S} \) is the total space of the determinant bundle on \( \text{Gr}(k, V) \), which is the top exterior power of \( E \) at every point \( E \in \text{Gr}(k, V) \).

Theorem. (Plucker relations) Let \( \tau \in \Lambda^k V \)

Then \( \tau \in \mathcal{S} \iff \sum \hat{\upsilon}_i \tau \otimes \hat{\upsilon}_i^* \tau = 0. \)

\[ \Lambda^{k+1} V \otimes \Lambda^{k-1} V \]

Proof. Let us first show that if \( \Sigma \hat{\upsilon}_i \tau \otimes \hat{\upsilon}_i^* \tau = 0 \). since the operator \( \Sigma \hat{\upsilon}_i \otimes \hat{\upsilon}_i^* \) is invariant under \( \text{GL}(V) \) action, it's enough to check this for \( \tau = \upsilon_1 \wedge \ldots \wedge \upsilon_k \). But this is easy: for any \( i \), either \( \hat{\upsilon}_i \) or \( \hat{\upsilon}_i^* \) kills \( \tau \).

Now let us prove the converse. let \( E(\tau) \) be the space of all \( \upsilon \in V \) such \( \hat{\upsilon}_i \tau = \upsilon \wedge \tau = 0 \) and \( E' \) be
the space of all \( f \in V^* \) such that \( f \circ \tau = i \circ \tau = 0 \). I claim that \( E \) and \( E' \) are orthogonal. Indeed, \( \hat{f} \circ \nu + \hat{\epsilon} \circ \hat{f} = f(\nu) \), so if \( \nu \in E \), \( \epsilon \in E' \) then \( f(\nu) = 0 \).

Thus \( E \subset E' \). Pick a basis \( \epsilon_i \) of \( V \) compatible with these subspaces. Let \( \dim E = m \), \( \dim E' = r \). So we have:

for \( S = \sum \epsilon_i \otimes \epsilon_i^* \):

\[
S(\tau \otimes \tau) = \sum_{i=1}^{m} \epsilon_i \tau \otimes \epsilon_i^* \tau + \sum_{i=m+1}^{n} \epsilon_i \tau \otimes \epsilon_i^* \tau + \sum_{i=r+1}^{n} \epsilon_i \tau \otimes \epsilon_i^* \tau.
\]

The first and the last sum are zero, as \( \epsilon_i \in E \), \( i \leq m \) and \( \epsilon_i^* \in E' \), \( i \geq r+1 \).

So \( \sum_{i=m+1}^{n} \epsilon_i \tau \otimes \epsilon_i^* \tau = 0 \).

But \( \epsilon_i \tau \) are linear independent for \( m+1 \leq i \leq n \), so \( \epsilon_i^* \tau = 0 \) for these \( i \).

But for these \( i \), \( \epsilon_i^* \in E' \), so we get that \( m = r \) and our sum is empty. The result is proved.
In coordinates: \( E \subset V = \mathbb{C}^n \) \( \dim E = m \)

\[ \begin{pmatrix}
  a_{11} & \cdots & a_{1m} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nm}
\end{pmatrix} \]

\( P_0(E) \in P \left( \binom{n}{k} \right) \) - minors of maximal size \( (m) \). For \( I \subseteq \{1, \ldots, n\} \), \( |I| = m \), have Plücker coordinate

\[ y_I = \det_{m \leq j \leq I} a_{ij} \]

\[ 1 \leq i \leq m \]

Prop. \[ S(\tau \otimes \tau) = 0 \iff \]

\( \forall I, J \subseteq \{1, \ldots, n\}, \ |I| = k-1, \ |J| = k+1 \)

\[ \sum_{j \in J \cup \{g\}} \prod_{j \notin I} P \left( \binom{n}{k} \right) \left( -1 \right) \]

where \( v_j \) is the number of \( j \) in \( J \) written in increasing order.

Pf. Exercise; this is just a rewriting of \( S(\tau \otimes \tau) = 0 \) in coordinates.
Now let's generalize to the infinite dimensional setting.

\[ F^{(0)} = \mathbb{V} = \mathbf{v}_0 \times \mathbf{v}_1 \times \ldots \]

**Definition.** \( \mathbb{V} = L_2(\mathbb{R}) \times \mathbb{R} \).

**Proposition.** \( \mathbf{v}_0 \times \mathbf{v}_1 \times \ldots \) for \( i_k + k = 0, k \geq 0 \) belongs to \( \mathbb{V} \).

**Proof.** We have a permutation

\[ \sigma : \mathbb{V} \rightarrow \mathbb{V}, \quad \sigma \in GL(\mathbb{R}), \]

which moves only finitely many elements \( s + \sigma(m) = \sigma(m), m \leq 0 \).

**Prop.** \( \tau \in \mathbb{V} \Longleftrightarrow \sum_{i \in \mathbb{Z}} \mathbf{v}_i \times \tau \otimes \mathbf{v}_i = 0 \).

(note that this sum is in fact finite).

**Proof.** Analogous to the finite dimensional case.

**Remark.** \( \mathbb{V} / \mathbb{V}_x = 16r \) is the infinite Grassmannian. It can be interpreted as the set of subspaces \( E \in C(l(t)) \) which contain \( x_k \mathbb{V} \) for \( k > 0 \), and...
dim $E/t^N \mathbb{C}[[t]] = N$.

Note that also $E \subseteq t^{-M} \mathbb{C}[[t]]$ for large enough $M$. So $Gr$ can be viewed as $U(Gr(N,N+M))$ or $U Gr(N,2N)_{n \geq 0}$ (exercise).

Now we would like to rewrite these infinite Plücker relations in terms of polynomials, using the Boson–Fermion Correspondence.