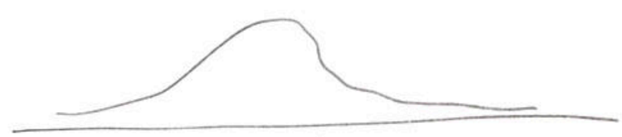
Lecture 10

Now I want to discuss the relationship of all this with integrable systems. One of the most important integrable PDEs is the KdV equation

$$u_t = \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}$$ (Coeff $\frac{3}{2}, \frac{1}{4}$ can be normalized to 1).

This was studied in 1895 by Korteweg and de Vries as an equation describing motion of waves in shallow water. One of the solutions is the Traveling wave solution:

$$f(t) = \frac{2u^2}{\cosh^2(ux+u^3t)}$$



Such solution is called a Soliton (solitary wave). It was first observed by J.S. Russell in 1834.

Generalization: Kadomtsev-Petviashvili equation

$$u_{yy} = \left(u_t - \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}\right)_x$$

This describes 2-dimensional waves.

We'll construct many solutions of these equations using repr. theory of ∞-dimensional Lie algebras. For this purpose we'll need infinite Grassmannian, so let us first recall about finite dimensional Grassmannians.

Let V be a finite dimensional vector space, with basis $v_1, \dots, v_n$. Then $GL(V)$ acts on $\Lambda^k V$, and the highest weight vector is $v_1 \wedge \dots \wedge v_k$. Denote by Ω the orbit of action of $GL(V)$ on $v_1 \wedge \dots \wedge v_k$. So Ω is the set of decomposable wedges:

<u>Prop.</u> $\Omega = \{ x \in \Lambda^k V, \exists x_1, \dots, x_k \in V, x = x_1 \wedge \dots \wedge x_k \}$.

<u>Pf. clear</u> (note that $x \neq 0$ implies that $x_1, \dots, x_k$ are linearly independent).

<u>Def</u>. The Grassmannian $Gr(k, V)$ is the set of all k-dim. subspaces of V.

Plücker embedding: $Pl: Gr(k, V) \Rightarrow \mathbb{P}\Lambda^k V$

$E \subset V$, $x_1, \dots, x_k$ basis $\Rightarrow Pl(E) = x_1 \wedge \dots \wedge x_k$.

<u>Exer.</u> Show that Pl is injective.

Clearly, $\mathrm{Im}(Pl) = \Omega/\mathbb{C}^*$

So $\Omega/\mathbb{C}^* \cong Gr(k,V)$.

(Ω is the total space of the determinant bundle on $Gr(k,V)$ which is the top exterior power of E at every point $E \in Gr(k,V)$.

<u>Theorem.</u> (Plücker relations) Let $\tau \in \Lambda^k V$, $\tau \neq 0$. Then $\tau \in \Omega \Leftrightarrow \sum \hat{v}_i \tau \otimes \check{v}_i^* \tau = 0$.

$$\cap$$

$$\Lambda^{k+1} V \otimes \Lambda^{k-1} V$$

<u>Proof.</u> Let us first show that if τ is a decomposable wedge then $\sum \hat{v}_i \tau \otimes \check{v}_i^* \tau = 0$. Since the operator $\sum \hat{v}_i \otimes \check{v}_i^*$ is invariant under $GL(V)$-action, it's enough to check this for $\tau = v_1 \wedge \dots \wedge v_k$. But this is easy: for any i, either $\hat{v}_i$ or $\check{v}_i^*$ kills τ.

Now let us prove the converse.

Let $E(\tau)$ be the space of all $v \in V$ such that $\hat{v}\tau = v \wedge \tau = 0$, and E' be the space of all $f \in V^*$ such that $\check{f}\tau = i_f \tau = 0$. I claim that E and E' are orthogonal. Indeed, $\check{f}\hat{v} + \hat{v}\check{f} = f(v)$, so if $v \in E$, $f \in E'$ then $f(v) = 0$.

Thus $E \subset E'^{\perp}$. Pick a basis e_i of V compatible with these subspaces. Let $\dim E = m$, $\dim E'^{\perp} = r$. So we have, for $S = \sum \hat{v}_i \otimes \check{v}_i^*$:

$$S(\tau \otimes \tau) = \sum_{i=1}^{m} e_i\tau \otimes e_i^*\tau + \sum_{i=m+1}^{r} e_i\tau \otimes e_i^*\tau +$$

$$+ \sum_{i=r+1}^{n} e_i\tau \otimes e_i^*\tau .$$

The first and the last sum are zero, as $e_i \in E$, $i \le m$, and $e_i^* \in E'$, $i \ge r+1$.

So $\sum_{i=m+1}^{r} e_i\tau \otimes e_i^*\tau = 0$.

But $e_i\tau$ are lin. independent for $m+1 \le i \le r$, so $e_i^*\tau = 0$ for these i. But for these i, $e_i^* \in E'$, so we get that $m = r$ and our sum is empty. The result is proved.

In coordinates: $E \subset V = \mathbb{C}^n$ $\dim E = m$

$m \times n$ matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

$Pl(E) \in \mathbb{P}^{\binom{m}{k}-1}$ - minors of maximal size (m). For $I \subset \{1, \dots, n\}$, $|I| = m$, have plücker coordinate

$$y_I = \det_{\substack{i \in I \\ 1 \le j \le m}} a_{ij}$$

Prop. $S(\tau \otimes \tau) = 0 \iff$

$\forall I, J \subset \{1, \dots, n\}$, $|I| = k-1$, $|J| = k+1$

$$\sum_{\substack{j \in J \\ j \notin I}} P_{I \cup \{j\}} P_{J \setminus \{j\}} (-1)^{\nu(j)} = 0$$

where $\nu(j)$ is the number of j in J written in increasing order.

Pf. Exercise; this is just a rewriting of $S(\tau \otimes \tau) = 0$ in coordinates.

Now let's generalize to the infinite dimensional setting.

$\mathcal{F}^{(0)} \ni \underline{\Psi} = v_0 \wedge \underline{v}_1 \wedge \dots$

<u>Definition</u>. $\Omega = GL(\infty) \cdot \underline{\Psi}$.

<u>Proposition</u>. $v_{i_0} \wedge v_{i_1} \wedge \dots$ for $i_k + k = 0$, $k \gg 0$, belongs to Ω.

<u>Pf</u>. We have a permutation

$\sigma : \mathbb{Z} \to \mathbb{Z}$, $\sigma \in GL(\infty)$,

which moves only finitely many elements s.t. $\sigma(m) = i_{-m}$, $m \leq 0$.

<u>Prop</u>. $\tau \in \Omega \Leftrightarrow \sum_{i \in \mathbb{Z}} \hat{v}_i \tau \otimes \check{v}_i^* \tau = 0$.

(note that this sum is in fact finite).

<u>Proof</u>. Analogous to the finite dimensional case.

<u>Remark</u>. $\Omega / \mathbb{C}^* = \mathrm{Gr}$ is the infinite Grassmannian. It can be interpreted as the set of subspaces $E \in \mathbb{C}((t))$ which contain $t^k \mathbb{C}[[t]]$ for $k \gg 0$, and

$\dim E/t^N\mathbb{C}[[t]] = N$.

Note that also $E \subset t^{-M}\mathbb{C}[[t]]$ for large enough M. So

Gr can be viewed as

$$\bigcup_{N,M}\left(Gr(N, N+M)\right), \text{ or } \bigcup_{N \geq 0} Gr(N, 2N)$$

(exercise).

Now we would like to rewrite these infinite Plücker relations in terms of polynomials, using the Boson – Fermion correspondence.