Infinite dimensional Lie algebras.

Lecture 1.

The goal of this course is to discuss the structure and representation theory of some of the most important infinite dimensional Lie algebras, and discuss the connections of this subjects with other fields, such as conformal field theory, the theory of integrable systems, and the theory of quantum groups.

Unlike the theory of finite dimensional Lie algebras, where there are powerful general theorems, such as the classification theorem for simple Lie algebras, there is no such luck in infinite dimensions: e.g., it is impossible to classify infinite dimensional simple

Lie algebras - it is a horrible mess. So we will study only some very special infinite dimensional Lie algebras which are especially important. Namely, we will mostly discuss the following Lie algebras:

1. The Heisenberg algebra (or oscillator algebra)
2. The Virasoro algebra (central extension of Witt algebra)
3. The Kac-Moody algebras. These contain simple finite dimensional Lie algebras, as well as affine Lie algebras, which are 1-dimensional central extensions of loop algebras $\mathfrak{g}[t,t^{-1}]$, where $\mathfrak{g}$ is a simple f.d. Lie algebra.

Let us now define some of these Lie algebras. All algebras will be over $\mathbb{C}$, unless specified otherwise.

<u>Definition</u> The oscillator algebra $\mathcal{A}$ is the Lie algebra $\mathcal{A} = \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}$ with commutator defined by the formula

$$[(f,\alpha),(g,\beta)] = \left(0,\ \operatorname{Res}_{t=0} g\,\overset{\overset{g f' dt}{\parallel}}{df}\right).$$

So $\mathcal{A}$ has a basis $\{a_n, n\in\mathbb{Z}; K\}$ with commutation relations

$$[a_n, K]=0,\quad [a_n,a_m] = n\delta_{n,-m}K.$$

(namely, $a_n=(t^n,0)$ and $K=(0,1)$). We see that $\mathcal{A}$ is a 1-dimensional central extension of the abelian Lie algebra with basis $\{a_n\}$.

Another example is the Lie algebra of vector fields. Namely, we have

<u>Definition.</u> The Witt algebra is the Lie algebra of polynomial vector fields on $\mathbb{C}^*$ (i.e, vector fields with $f(t)\partial$, where $f\in\mathbb{C}[t,t^{-1}]$), with bracket being commutator of vector fields:

$$[f\partial, g\partial] = (fg'-gf')\partial \quad \left(\text{here } \partial = \frac{d}{dt}\right)$$

So W has a basis $\{L_n, n\in\mathbb{Z}\}$ with commutator

$$[L_n,L_m] = (n-m)L_{n+m}$$

Namely, $L_n = -t^{n+1}\partial$

<u>Remark</u>. In infinite dimensions, a Lie algebra does not always correspond to a Lie group. For example, there is no Lie group corresponding to W. There is only a Lie group that in some sense corresponds to the real form $W_{\mathbb{R}}$ of W, consisting of all the vector fields in W which are tangent to the unit circle in $\mathbb{C}$. Such vector fields are of the form $\varphi(\theta)\frac{d}{d\theta}$, where φ is a trigonometric polynomial. Since $t = e^{i\theta}$, we have $\frac{\partial}{\partial\theta} = it\frac{d}{dt}$. So $f(t)\partial \in W_{\mathbb{R}}$ iff f takes imaginary values on $|t| = 1$, i.e. $f(t) = \sum \alpha_n t^n$ with $\alpha_n = -\overline{\alpha_{-n}}$.

Namely, the group corresponding to $W_{\mathbb{R}}$ is the group of diffeomorphisms of the circle, $\mathrm{Diff}\, S^1$. More precisely, $\mathrm{Lie}\, \mathrm{Diff}\, S^1$, in an appropriate sense, is some completion of $W_{\mathbb{R}}$

namely the Lie algebra $\widehat{W}_{\mathbb{R}}$ of all smooth vector fields $\varphi(\theta)\frac{d}{d\theta}$, $\varphi \in C^\infty(S^1)$ (not necessarily a trigonometric polynomial, but any trigonometric, i.e. Fourier, series with rapidly decaying coefficients). This is because if $\theta \mapsto g_s(\theta)$, $\theta \mapsto h_u(\theta)$ are smooth families of diffeomorphisms of S^1 with $h_0 = g_0 = id$, $\left.\frac{\partial g}{\partial s}\right|_{s=0} = \varphi$, $\left.\frac{\partial h}{\partial u}\right|_{u=0} = \psi$, then $\left.\frac{\partial^2}{\partial s \partial u}\right|_{s=u=0} g_s \circ h_u \circ g_s^{-1} \circ h_u^{-1}$ $= \varphi\psi' - \psi\varphi'$ which corresponds to the Lie bracket in $\widehat{W}_R$ (exercise). But the group $\mathrm{Diff}\, S^1$ does not have a complexification. The best thing you can say is that there is a semigroup of annuli (defined by G. Segal and playing an important role in conformal field theory) whose "Lie algebra" in some sense is W (or rather its completion $\hat{W}$, the Lie algebra of complex vector fields on S^1). But this semigroup cannot be embedded into a group.

Still we should always heuristically think about Lie algebra symmetry as an infinitesimal version of group symmetry, even in infinite dimensions.

For instance, we have the following lemma.

<u>Lemma</u>. We have a natural homomorphism $\eta : W \to \mathrm{Der}\,\mathcal{A}$ given by $\eta(f\partial)(g,\alpha) = (fg', 0)$.

<u>Proof</u>. The Lemma is obvious, since residue of a differential 1-form is invariant under charges of variable ($\mathrm{Res}_{t=0}\, g_2 dg_1 = \frac{1}{2\pi i}\oint_{|t|=1} g_2(t)\, dg_1(t)$ which is invariant under diffeomorphisms of $S^1 = \{t \mid |t|=1\}$). Also it is easy to check by direct calculation (exercise).

In applications, one encounters representations not quite of W but rather of its universal central extension, which is called the Virasoro algebra. This is a 1-dimensional central extension, which is obtained by adding to W the 1-dimensional center $\mathbb{C}$. We would like to derive a formula for this extension, and show that it is the unique nontrivial extension and that it is universal. To this end, let us recall the theory of central algebra extensions

Suppose that L is a Lie algebra, and $\hat{L}$ is a 1-dimensional central extension of L: $0 \to \mathbb{C} \to \hat{L} \to L \to 0$. This means that $\hat{L}$ admits with a splitting $\hat{L} = L \oplus \mathbb{C}$ of vector spaces, such that the Lie bracket in $\hat{L}$ is defined by the formula

$$[(a,\alpha),(b,\beta)] = ([ab], \omega(a,b)),$$

where $\omega : \Lambda^2 L \to \mathbb{C}$ is a skew-symmetric bilinear form. The Jacobi identity for $\hat{L}$ implies that ω has to satisfy the 2-cocycle condition:

$$\omega([ab],c) + \omega([bc],a) + \omega([ca],b) = 0,$$

and conversely, any ω that satisfies the 2-cocycle condition defines an extension. On the other hand, the splitting $\hat{L} = L \oplus \mathbb{C}$ is not canonical, so the same extension may correspond to different forms ω. Indeed, let $\hat{L}_{\omega_1}$ and $\hat{L}_{\omega_2}$ be two extensions. An isomorphism of extensions would be a map $\hat{L}_{\omega_1} \xrightarrow{\hat{\xi}} \hat{L}_{\omega_2}$ given by

$\hat{\xi}(a, \alpha) = (a, \alpha + \xi(a))$, where $\xi \in L^*$.
The condition on ξ is then

$$\hat{\xi}\left([(a,\alpha),(b,\beta)]_1\right) = [\hat{\xi}(a,\alpha), \hat{\xi}(b,\beta)]_2$$

i.e. $\xi([a,b]) = \omega_2(a,b) - \omega_1(a,b)$

Thus, extensions up to an isomorphism are parametrized by Z^2/B^2, where Z^2 is the space of 2-cocycles and B^2 is the space of forms $\xi([ab])$ (2-coboundaries). This space is denoted by $H^2(L)$ and is called the 2-nd cohomology of L (One can define cohomology $H^i(L)$ for all $i \geq 0$; e.g. $H^0(L) = \mathbb{C}$ and $H^1(L) = \mathrm{Hom}_{Lie}(L, \mathbb{C})$; but we will not discuss this in detail).

Now we will calculate $H^2(W)$.

<u>Theorem</u>. The space $H^2(W)$ is 1-dimensional, spanned by the element ω given by

$$\omega(L_n, L_m) = (n^3 - n)\delta_{n,-m}.$$

<u>Proof</u>. Let $\beta \in Z^2(W)$. Pick a linear functional $\xi \in W^*$ such that

$\xi(L_n) = \frac{1}{n}\beta(L_n, L_0)$, $n \neq 0$, and replace β by $\tilde{\beta}$, defined by $\tilde{\beta}(a,b) = \beta(a,b) - \xi([a,b])$. By doing this, we may assume that $\beta(L_n, L_0) = 0 \quad \forall n$.

Now,

$$\beta([L_0, L_m], L_n) + \beta([L_n, L_0], L_m) + \beta([L_m, L_n], L_0) = 0, \text{ so}$$

$$(n+m)\beta(L_n, L_m) = 0.$$

Thus, $\beta(L_n, L_m) = b_n \delta_{n,-m}$, where $b_n \in \mathbb{C}$. Our job is to find b_n. Clearly, $b_{-n} = -b_n$.

Let $m+n+p=0$; then

$\beta([L_m, L_n], L_p) = (n-m) b_p$

So we have

$$(n-m) b_p + (m-p) b_n + (p-n) b_m = 0,$$

or

$$(n-m) b_{n+m} = (2m+n) b_n - (2n+m) b_m.$$

In particular, for $m=1$,

$$(n-1) b_{n+1} = (n+2) b_n - (2n+1) b_1.$$

By replacing β with $\hat{\beta}(a,b) = \beta(a,b) - \frac{b_1}{2}\xi_0([a,b])$ where ξ_0 is the L_0-coefficient, we may assume that $b_1 = 0$. Then

$$(n-1) b_{n+1} = (n+2) b_n, \text{ or } (n-2) b_n = (n+1) b_{n-1}.$$

So

$$b_n = \frac{(n+1) n (n-1)(n-2) \cdots 4}{(n-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1} b_2 = \frac{n^3 - n}{6} b_2.$$

This proves the theorem.

<u>Definition</u>. The Virasoro algebra Vir is the central extension

of W defined by the 2-cocycle
$$\omega(L_n, L_m) = \frac{n^3-n}{12}\delta_{n,-m}.$$
Thus, the basis of Vir is $\{L_n\}_{n\in\mathbb{Z}}$ and C, with
$$[L_n, C]=0, \quad [L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}C$$
The factor $\frac{1}{2}$ is just a normalization, which is equivalent to $b_1=0$, $b_2=\frac{1}{2}$. It will become clear later why such normalization is chosen.

Now let us consider the case of loop algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with an invariant symmetric bilinear form $(,)$. In this case, we can define the following 1-cocycle on $\mathfrak{g}[t,t^{-1}]$:
$$\omega(f,g) = \operatorname{Res}_{t=0}(df, g).$$
So if $f=\sum f_i t^i$, $g=\sum g_i t^i$ then $\omega(f,g)=\sum i f_i g_{-i}$.

It is easy to check that it is a 2-cocycle (exercise), so it defines a 1-dimensional central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}[t,t^{-1}]$ with bracket

$$[(f,\alpha),(g,\beta)] = \left([f,g], \operatorname{Res}_{t=0}(df,g)\right).$$

Note that if $\mathfrak{g} = \mathbb{C}$ and $(a,b) = ab$, we get the Heisenberg algebra.

<u>Theorem</u>. If $\mathfrak{g}$ is simple then $H^2(\mathfrak{g}[t,t^{-1}]) = \mathbb{C}$, spanned by the cocycle ω corresponding to the invariant inner product on $\mathfrak{g}$. So in this case the above 1-dimensional central extension is the unique nontrivial one.

<u>Proof</u>. Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. Recall that $Z^1(\mathfrak{g},M)$ (1-cocycles) is the space of $\eta : \mathfrak{g} \to M$ such that $\eta([ab]) = a\circ\eta(b) - b\circ\eta(a)$, and $B^1(\mathfrak{g},M) \subset Z^1(\mathfrak{g},M)$ is the subspace of $\eta(a) = a\circ m$, $m \in M$. Also, $H^1(\mathfrak{g},M) = Z^1(\mathfrak{g},M)/B^1(\mathfrak{g},M)$ is the first cohomology of $\mathfrak{g}$ with coefficients in M

We have $H^1(\mathfrak{g}, M) = \mathrm{Ext}^1_{\mathfrak{g}}(\mathbb{C}, M)$ (and more generally $\mathrm{Ext}^1_{\mathfrak{g}}(N, M) = H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{C}}(N, M))$)

<u>Theorem</u> (Whitehead) If $\mathfrak{g}$ is fin. dim simple and M is f. dim then $H^1(\mathfrak{g}, M) = 0$.

<u>Proof</u>. Every finite dim. representation of $\mathfrak{g}$ is completely reducible

Also, if $\omega \in Z^2(\mathfrak{g})$, $\mathfrak{g}_0 \subset \mathfrak{g}$ a Lie subalgebra, and $M \subset \mathfrak{g}$ a $\mathfrak{g}_0$-submodule then we can view ω as an element of $Z^1(\mathfrak{g}_0, M^*)$

Now let $\mathfrak{g}$ be finite dimensional simple, and ω be a 2-cocycle on $\mathfrak{g}[t, t^{-1}]$. Consider the restriction of ω to $\mathfrak{g} \times \mathfrak{g}t^n$. It belongs to $Z^1(\mathfrak{g}, \mathfrak{g}^*)$. Since $H^1(\mathfrak{g}, \mathfrak{g}^*) = 0$, it is in $B^1(\mathfrak{g}, \mathfrak{g}^*)$, so there is $\xi_n : \mathfrak{g}t^n \to \mathbb{C}$ such that $\omega(a, bt^n) = \xi_n([ab]t^n)$, $a, b \in \mathfrak{g}$.

Let $\xi : \mathfrak{g}[t, t^{-1}] \to \mathbb{C}$, $\xi|_{\mathfrak{g}t^n} = \xi_n$, and replace $\omega(a,b)$ with $\omega(a,b) - \xi([ab])$. Then

$\omega(a, bt^n) = 0 \quad \forall a \in \mathfrak{g}, b \in \mathfrak{g}$.

So $\omega([ab]t^n, ct^m) + \omega(bt^n, [ac]t^m)$

$= 0$, $a, b, c \in \mathfrak{g}$. Thus the map

$(b \otimes c) \longmapsto \omega(bt^n, ct^m)$ is an invariant in $\mathfrak{g}^* \otimes \mathfrak{g}^*$, so

$$\omega(bt^n, ct^m) = \gamma_{n,m} \, (\, , \,), \quad \gamma_{n,m} = -\gamma_{m,n} \in \mathbb{C}.$$

We have

$$\gamma_{n,m+p} + \gamma_{m,p+n} + \gamma_{p,m+n} = 0.$$

So for each s,

$$\gamma_{n,s-n} + \gamma_{m,s-m} = \gamma_{n+m,s-n-m}, \text{ and } \gamma_{0,s} = 0.$$

Thus $\gamma_{n,s-n} = n\gamma_s$, where $\gamma_s = \gamma_{1,s-1}$

So $(s-n)\gamma_s = -n\gamma_s \Rightarrow s\gamma_s = 0 \Rightarrow \gamma_s = 0$ for $s \neq 0$.

Now for $s=0$, we have

$$\gamma_{n,-n} = n\gamma\ , \quad \gamma = \gamma_{1,-1},$$

so we get a 1-dimensional space of solutions (as we no longer have freedom of adding ξ). The theorem is proved.

<u>Def.</u> If $\mathfrak{g}$ is simple finite dimensional, $\hat{\mathfrak{g}}$ is called an (untwisted) affine Kac-Moody algebra.