18.705 final exam
Tuesday, December 15, 2015

NAME:

You may use your class notes, homeworks, and Altman-Kleiman notes in a printed form. You may use statements from the notes without proof.

You **cannot** use electronic equipment (except watches) during the exam.

1. (10 points) Let $R = \mathbb{C}[x,y]/\langle x^2 - y^3, 4y^2 - 5x^2y + x^4\rangle$.

(i) Show that R is Artinian, and find the maximal ideals in R. How many maximal ideals are there?

(ii) Compute the dimension of the localization $R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R, and the dimension of R as complex vector spaces.

Solution. (i) Maximal ideals correspond to solutions of the equations

$$x^2 - y^3 = 0, 4y^2 - 5x^2y + x^4 = 0.$$

Substituting the first equation in the second one, we get $4y^2 - 5y^4 + y^6 = 0$. Thus $y = 0$ or $y^4 - 5y^2 + 4 = 0$, so $y = \pm 1$ or $y = \pm 2$, where for each $y \neq 0$, we have two values of x differing by sign, and for $y = 0$ we have $x = 0$, so we get 9 maximal ideals. Thus R is Artinian.

(ii) If $y \neq 0$, then $R_{\mathfrak{m}} = \mathbb{C}$, so the dimension is 1. So consider $\mathfrak{m} = (x,y)$. In this case in $R_{\mathfrak{m}}$, x and y are nilpotent. So $y^2 = 0$ and hence $x^2 = 0$. On the other hand, modulo $\mathfrak{m}^3$, the relations are $x^2 = 0, y^2 = 0$, so $xy \neq 0$, and we get $\dim R_{\mathfrak{m}} = 4$. Thus $\dim R = 12$.

2. (10 points) (i) Let f be a polynomial with complex coefficients of degree n. Describe the possible rings of the form $\mathbb{C}[x]/\langle f\rangle$. How many different rings up to isomorphism can you get for $n = 5$?

(ii) Let f be a polynomial with real coefficients of degree n. What are the possible values of the length of $\mathbb{R}[x]/\langle f\rangle$ as a module over itself?

(iii) How many different rings up to isomorphism can you get in (ii) for $n = 4$?

Solution. (i) The rings correspond to types of factorization of f: $R = \oplus_i \mathbb{C}[t]/t^{m_i}$, where m_i are the multiplicities of the linear factors of f. Thus the number of possible rings is $p(n)$, the number of partitions of n. So for $n = 5$ we get 7 options.

(ii) Suppose f has $2r$ non-real roots and $n - 2r$ real roots. Then the length is $r+(n-2r) = n-r$. So the possible values of the length are all integers $\ell \in [n/2, n]$.

(iii) The type of the ring is determined by the multiplicities of the quadratic and linear factors in the irreducible decomposition. So the answer for a general n is $\sum_r p(r)p(n-2r)$, where $p(r)$ is the number of partitions of r. So for $n = 4$ we get $p(4) + p(2) + p(2) = 5 + 2 + 2 = 9$.

3. (2 points each question) Are the following statements true or false? (Mark T/F without proof).

T (a) A quotient ring of $\mathbb{C}[x_1, ..., x_n]$ which has finitely many maximal ideals is finite dimensional over $\mathbb{C}$.

F (b) The sum of any two prime ideals in $\mathbb{C}[x, y]$ is prime.

F (c) If M is a flat R-module then the functor $\mathrm{Hom}(M, ?)$ is exact.

T (d) Let R be the ring of rational numbers given by fractions with odd denominators. If M, N are finitely generated modules over R and $M \otimes N = 0$ then $M = 0$ or $N = 0$.

T (e) If the total quotient ring of a ring R is reduced (i.e., has no nonzero nilpotent elements) then R is reduced.

F (f) An R-algebra R' is module-finite over R if and only if for every maximal ideal $\mathfrak{m}$ in R, $R'/\mathfrak{m}R'$ is a finite dimensional vector space over $R/\mathfrak{m}$.

F (g) If M is a $\mathbb{C}[x]$-module and $M/\mathfrak{m}M = 0$ for all maximal ideals $\mathfrak{m}$ in $\mathbb{C}[x]$ then $M = 0$.

T (h) Any finitely generated projective module over $\mathbb{C}[[x_1, ..., x_n]]/I$ is free (where $I \subset \mathbb{C}[[x_1, ..., x_n]]$ is an ideal).

T (i) Any $\mathbb{C}$-subalgebra of $\mathbb{C}[x]$ is Noetherian.

T (j) Let $f \in \mathbb{C}[x_1, ..., x_n]$. Then the principal ideal $\langle f \rangle$ has a *unique* irredundant primary decomposition.

T (k) The associated primes of $\langle f \rangle$ in (j) are generated by the irreducible factors of f.

F (l) In (j), $\langle f \rangle$ is the intersection of its associated primes.

F (m) The ring $R = \mathbb{C}[x, y]/\langle y^2 - x^3 - x \rangle$ is a principal ideal domain.

T (n) The localization of the ring R from (m) at any maximal ideal is a principal ideal domain.

F (o) The localization of $R = \mathbb{C}[x, y]/\langle y^2 - x^3 - x^2 \rangle$ at any maximal ideal is a principal ideal domain.

4. (10 points) Compute the Hilbert series of the ring $\mathbb{C}[x,y,z]/\langle x^2-z^2, y^2-z^2\rangle$ (as a rational function of t), assuming that x, y, z have degree 1.

Solution. Using the relations, it is easy to express all monomials in terms of z^i, xz^i, yz^i, xyz^i. We claim that these monomials are linearly independent. Indeed, consider any linear relation between them, and specialize it at some $z=a$ so that the specialization is nontrivial. Then we will get that the algebra $\mathbb{C}[x,y]/(x^2-a^2, y^2-a^2)$ has dimension <4, which is a contradiction.

Thus the Hilbert series is $\frac{(1+t)^2}{1-t}$.

5. (10 points) Let I be the ideal in $\mathbb{C}[x, y]$ consisting of polynomials that vanish at $(0,0)$, $(0,1)$, and $(1,0)$. Find a system of two generators for I.

Solution. We can take generators $u = xy$ and $v = (y - x)(1 - x - y)$. Indeed, these polynomials are contained in I. Also $\mathbb{C}[x, y]/(xy)$ is the ring of pairs $(a(x), b(y))$ of polynomials such that $a(0) = b(0)$, and the image of I is the subset of pairs $(x(x-1)p(x), y(y-1)q(y))$, while the second generator maps to $(x(x-1), -y(y-1))$, so it generates exactly the image of I.

6. (10 points) (i) Find the associated primes of the ideal $I = \langle y^2, x(x-1)y \rangle$ in $\mathbb{C}[x, y]$.

(ii) Find an irrredundant primary decomposition of this ideal.

Solution. (i) Consider $R = \mathbb{C}[x, y]/I$. Then R has a basis $x^i, i \geq 0$, y and xy. Let us compute the annihilator of an element $v \in R$. If the image $v_0(x)$ of v in $\mathbb{C}[x]$ is nonzero, then the annihilator is contained in (y) and contains (y^2), and equals (y) iff v_0 is divisible by $x(x-1)$. So this gives the associated prime (y). Now suppose $v = (a + bx)y$. Its annihilator contains $(x, y) \cap (x-1, y)$, and is one of the summands if $v = xy$ or $v = (x-1)y$. So we also get associated primes (x, y) and $(x-1, y)$.

(ii) One can take $I = (y) \cap (x, y^2) \cap (x-1, y^2)$.

1. (Each question is 3 points; each mistake or omission is -1 point; if you have more than one mistake in one question, you get 0 for this question)

Consider the following rings:

(1) $\mathbb{Z}[1/6]$

(2) The ring R of rational numbers which can be written as a fraction with denominator not divisible by 3.

(3) $\mathbb{Z}/10$

(4) $\mathbb{Z}/9$

(5) $\mathbb{Z}/11$

(6) The localization $\mathbb{C}[x,y]_{\langle x,y\rangle}$ of $\mathbb{C}[x]$ at $\langle x,y\rangle$.

(7) $\mathbb{Z}[1/2,x]$

(8) $\mathbb{Q}[x]/\langle x^3-2x^2+x\rangle$

(9) $\mathbb{Q}[x]/\langle x^2-2\rangle$

(10) $\mathbb{Q}[x]/\langle x^2-x-2\rangle$

(11) $\mathbb{R}[x]/\langle x^2-2\rangle$

(12) $\mathbb{Z}[X_1,X_2,...]$ (infinitely many variables)

List (without proof) the item numbers of ALL the rings above which are:

(a) Have Krull dimension 1: 1, 2

(b) Artinian rings: 3, 4, 5, 8, 9, 10, 11

(c) principal ideal domains: 1, 2, 5, 9

(d) Noetherian rings: all but 12

(e) local rings: 2, 4, 5, 6, 9

(f) finitely generated rings (as $\mathbb{Z}$-algebras): 1, 3, 4, 5, 7

(g) finitely generated $\mathbb{Z}$-modules: 3, 4, 5

(h) reduced rings: 1, 2, 3, 5, 6, 7, 9, 10, 11, 12

(i) flat $\mathbb{Z}$-modules: 1, 2, 6, 7, 8, 9, 10, 11, 12

(j) free $\mathbb{Z}$-modules: 12