18.705 midterm exam
Thursday, October 22, 2015, 1.05-2.25pm

NAME:

You may use your notes, homeworks, and Altman-Kleiman notes in a printed form. You may use statements from the notes without proof.

You **cannot** use electronic equipment (except watches) during the exam.

1. (Each question is 3 points; each mistake or omission is -2 points; if you have more than one mistake in one question, you get 0 for this question)

Consider the following rings:

(1) $\mathbb{Z}[1/2]$

(2) The ring R of rational numbers which can be written as a fraction with odd denominator.

(3) $\mathbb{Z}/2$

(4) $\mathbb{Z}/4$

(5) $\mathbb{Z}/6$

(6) $\mathbb{Z}/2 \times \mathbb{Z}/2$

(7) $\mathbb{Z}[\sqrt{-5}]$

(8) $\mathbb{Z}[x]$

(9) $\mathbb{Q}[x]/ < x^2 + 1 >$

(10) $\mathbb{Q}[x]/ < x^2 - 1 >$

(11) $\mathbb{Q}[x]/ < x^2(x-1) >$

(12) $\mathbb{Z}[[x]]$

List (without proof) the item numbers of ALL the rings above which are:

(a) domains: 1, 2, 3, 7, 8, 9, 12

(b) unique factorization domains: 1, 2, 3, 8, 9, 12

(c) principal ideal domains: 1, 2, 3, 9

(d) fields: 3, 9

(e) local rings: 2, 3, 4, 9

(f) finitely generated rings (as $\mathbb{Z}$-algebras): 1, 3, 4, 5, 6, 7, 8

(g) finitely generated $\mathbb{Z}$-modules: 3, 4, 5, 6, 7

(h) reduced rings: 1, 2, 3, 5, 6, 7, 8, 9, 10, 12

(i) flat $\mathbb{Z}$-modules: 1, 2, 7, 8, 12, 9, 10, 11

(j) free $\mathbb{Z}$-modules: 7, 8

2. (Each question is 3 points; each mistake or omission is -2 points; if you have more than one mistake in one question, you get 0 for this question)

Consider the following ideals in rings.

(1) $R = \mathbb{Z}$, $I =< 8, 12 >$

(2) $R = \mathbb{Z}[x]$, $I =< 2, x^2 + x + 1 >$

(3) $R = \mathbb{Z}[x]$, $I =< 2, x^2 + 1 >$

(4) $R = \mathbb{Z}[x]$, $I =< 2, x^2 + x >$

(5) $R = \mathbb{Q}[x, y]$, $I =< x, y >$

(6) $R = \mathbb{Z}/8$, $I =< 4 >$

(7) $R = \mathbb{Q}[x_1, x_2, ...]$, $I =< x_1, x_2, ... >$ (infinitely many variables)

(8) $R = \mathbb{Z} \times \mathbb{Z}$, $I = \{(n, 0), n \in \mathbb{Z}\}$

(9) R is the ring of rational functions $f \in \mathbb{C}(x)$ regular at 0, $I =< x >$

(10) $R = \mathbb{C}[[x]]$, $I =< x(x + 1) >$

List (without proof) the item numbers of ALL the ideals above which are:

(a) prime: 2, 5, 7, 8, 9, 10

(b) maximal: 2, 5, 7, 9, 10

(c) principal: 1, 6, 8, 9, 10

(d) contained in the nilpotent radical nil(R): 6

(e) radical (coincide with their radicals): 2, 4, 5, 7, 8, 9, 10

(f) projective R-modules: 1, 8, 9, 10

(g) finitely generated R-modules: all but 7

(h) free R-modules: 1, 9, 10

(i) flat R-modules: 1, 8, 9, 10

(j) contained in the Jacobson radical rad(R): 6, 9, 10

3. (a) (10 points) Find the order of the group $A := (\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.
(b) (10 points) Compute the dimension of the $\mathbb{C}$-vector space

$$V := (\mathbb{C}[x,y]/ < xy >) \otimes_{\mathbb{C}[x,y]} (\mathbb{C}[x,y]/ < y - x^2 >).$$

Justify your answers.

(a) $A = 0$, $|A| = 1$. Indeed, for $r, p \in \mathbb{Z}$, $s, q \in \mathbb{N}$

$$\frac{r}{s} \otimes \frac{p}{q} = \frac{r}{sq} \cdot q \otimes \frac{p}{q} = \frac{r}{sq} \otimes q\frac{p}{q} = \frac{r}{sq} \otimes p = 0$$

(b) $V = \mathbb{C}[x,y]/\langle xy, y-x^2 \rangle$ because of the folowing lemma.

<u>Lemma</u> If M is an R-module, $\mathfrak{a} < R$ an ideal then $M \otimes_R R/\mathfrak{a} = M/\mathfrak{a}M$.

<u>Pf.</u> $M \otimes_R R/\mathfrak{a}$ is spanned by $m \otimes 1$, and the relations are $am \otimes 1 = m \otimes a = 0$, $a \in \mathfrak{a}$. $\square$

Now, if $xy = 0$ and $y - x^2 = 0$ then $x^3 = 0$, $y = x^3$, so $\mathbb{C}[x,y]/\langle xy, y-x^2 \rangle = \mathbb{C}[x]/x^3$, so $\boxed{\dim V = 3}$

4. (5 points each question) (a) For which positive integer n does there exist a maximal ideal in $\mathbb{Q}[x]$ of codimension n over $\mathbb{Q}$?

(b) For which positive integer n does there exist a maximal ideal in $\mathbb{R}[x]$ of codimension n over $\mathbb{R}$?

(c) Describe maximal ideals in $\mathbb{R}[x, y]$.

(d) Describe maximal ideals in $\mathbb{Z}[i]$, where $i^2 = -1$. (Hint: show and use that -1 is a square modulo a prime p iff $p = 1 \bmod 4$). or $p=2$

Justify your answers.

(a) Any n. Can take $m = \langle x^n - 2 \rangle$.
This is maximal since $\mathbb{Q}[x]/\langle x^n-2\rangle$ is a field.

(b) $n = 1$ or 2. The quotient must be a finite field extension of $\mathbb{R}$, so it is $\mathbb{R}$ or $\mathbb{C}$.

Ex: $\mathbb{R}[x]/\langle x^2+1\rangle = \mathbb{C}$, $\mathbb{R}[x]/\langle x\rangle = \mathbb{R}$.

(c) 1. $\forall\, x_0, y_0 \in \mathbb{R}$, $m = \{f \mid f(x_0, y_0) = 0\}$.

2. $\forall$ pair of complex conjugate points $(x_0, y_0) \neq (\bar{x}_0, \bar{y}_0)$, $m = \{f \mid f(x_0, y_0) = f(\bar{x}_0, \bar{y}_0) = 0\}$.

(d) Let $m \subset \mathbb{Z}[i]$ be a maximal ideal.
Then $\exists x \in m$, $x \neq 0$ (as $\mathbb{Z}[i]$ is not a field).
Then $n = x\bar{x} \in m$, $n \in \mathbb{N}$. So $n = 0$ in $F := \mathbb{Z}[i]/m$.
Hence F is a field of characteristic $p > 0$, and we have a surjection $\mathbb{F}_p[i] \twoheadrightarrow F$, where $i = \sqrt{-1}$.
If $\exists \sqrt{-1} \in \mathbb{F}_p$ ($\Leftrightarrow p = 1 \bmod 4$ or $p = 2$) then $F = \mathbb{F}_p$,
so $m = \langle p, i - r\rangle$, where $r^2 = -1 \bmod p$. (in fact, this ideal is principal as $\mathbb{Z}[i]$ is a PID, but it does not matter).
If $\sqrt{-1} \notin \mathbb{F}_p$ ($\Leftrightarrow p \neq 1 \bmod 4$ and $p \neq 2$)
then $m = \langle p \rangle$.

5. (10 points) Let $R = \mathbb{C}[x, y]$, $R' = \mathbb{C}[[x, y]] \supset R$. Give an example of a prime ideal $\mathfrak{p} \subset R$ such that $\mathfrak{p}R' \neq R'$ but $\mathfrak{p}R'$ is NOT a prime ideal in R'.

Take $\mathfrak{p} = \langle \underbrace{x^2(x+1) - y^2}_{f} \rangle$. Since f is irreducible, $\mathfrak{p}$ is prime. But f splits in $\mathbb{C}[[x,y]]$:

$$f = (x\sqrt{x+1} - y)(x\sqrt{x+1} + y),$$

where $\sqrt{x+1}$ is the Taylor series of the function $\sqrt{x+1}$.

Hence, $\mathfrak{p}R'$ is not a prime ideal in R' (and $\mathfrak{p}R' \neq R'$).