18.705 midterm exam
Thursday, October 22, 2015, 1.05-2.25pm

NAME:

You may use your notes, homeworks, and Altmann-Kleiman notes in a printed form. You may use statements from the notes without proof.

You cannot use electronic equipment (except watches) during the exam.

1. (Each question is 3 points; each mistake or omission is -2 points; if you have more than one mistake in one question, you get 0 for this question)

Consider the following rings:

(1) \( \mathbb{Z}[1/2] \)

(2) The ring \( R \) of rational numbers which can be written as a fraction with odd denominator.

(3) \( \mathbb{Z}/2 \)

(4) \( \mathbb{Z}/4 \)

(5) \( \mathbb{Z}/6 \)

(6) \( \mathbb{Z}/2 \times \mathbb{Z}/2 \)

(7) \( \mathbb{Z}[\sqrt{-5}] \)

(8) \( \mathbb{Z}[x] \)

(9) \( \mathbb{Q}[x]/\langle x^2 + 1 \rangle \)

(10) \( \mathbb{Q}[x]/\langle x^2 - 1 \rangle \)

(11) \( \mathbb{Q}[x]/\langle x^2(x-1) \rangle \)

(12) \( \mathbb{Z}[x] \)

List (without proof) the item numbers of ALL the rings above which are:

(a) domains: \( 1, 2, 3, 4, 5, 6, 7, 8, 9, 12 \)

(b) unique factorization domains: \( 1, 2, 3, 8, 9, 12 \)

(c) principal ideal domains: \( 1, 2, 3, 9 \)

(d) fields: \( 3, 9 \)

(e) local rings: \( 2, 3, 4, 9 \)

(f) finitely generated rings (as \( \mathbb{Z} \)-algebras): \( 1, 3, 4, 5, 6, 7, 8 \)

(g) finitely generated \( \mathbb{Z} \)-modules: \( 3, 4, 5, 6, 7 \)

(h) reduced rings: \( 1, 2, 3, 5, 6, 7, 8, 9, 10, 12 \)

(i) flat \( \mathbb{Z} \)-modules: \( 1, 2, 7, 8, 12, 9, 10, 11 \)

(j) free \( \mathbb{Z} \)-modules: \( 7, 8 \)
2. (Each question is 3 points; each mistake or omission is -2 points; if you have more than one mistake in one question, you get 0 for this question)

Consider the following ideals in rings.

(1) $R = \mathbb{Z}$, $I = \langle 8, 12 \rangle$

(2) $R = \mathbb{Z}[x]$, $I = \langle 2, x^2 + x + 1 \rangle$

(3) $R = \mathbb{Z}[x]$, $I = \langle 2, x^2 + 1 \rangle$

(4) $R = \mathbb{Z}[x]$, $I = \langle 2, x^2 + x \rangle$

(5) $R = \mathbb{Q}[x, y]$, $I = \langle x, y \rangle$

(6) $R = \mathbb{Z}/8$, $I = \langle 4 \rangle$

(7) $R = \mathbb{Q}[x_1, x_2, ...]$, $I = \langle x_1, x_2, ... \rangle$ (infinitely many variables)

(8) $R = \mathbb{Z} \times \mathbb{Z}$, $I = \{(n, 0), n \in \mathbb{Z}\}$

(9) $R$ is the ring of rational functions $f \in \mathbb{C}(x)$ regular at 0, $I = \langle x \rangle$

(10) $R = \mathbb{C}[x]$, $I = \langle x(x + 1) \rangle$

List (without proof) the item numbers of ALL the ideals above which are:

- (a) prime: $2, 5, 7, 8, 9, 10$
- (b) maximal: $2, 5, 7, 9, 10$
- (c) principal: $1, 6, 8, 9, 10$
- (d) contained in the nilpotent radical nil($R$): $6$
- (e) radical (coincide with their radicals): $2, 4, 5, 7, 8, 9, 10$
- (f) projective $R$-modules: $1, 8, 9, 10$
- (g) finitely generated $R$-modules: all but 7
- (h) free $R$-modules: $1, 9, 10$
- (i) flat $R$-modules: $1, 8, 9, 10$
- (j) contained in the Jacobson radical rad($R$): $6, 9, 10$
3. (a) (10 points) Find the order of the group \( A := (\mathbb{Q}/\mathbb{Z}) \otimes _{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \).

(b) (10 points) Compute the dimension of the \( \mathbb{C} \)-vector space 
\[ V := (\mathbb{C}[x,y]/\langle xy \rangle) \otimes _{\mathbb{C}[x,y]} (\mathbb{C}[x,y]/\langle y - x^2 \rangle). \]

Justify your answers.

(a) \( A = 0 \), \( |A| = 1 \). Indeed, for \( r, p \in \mathbb{Z} \), \( s, q \in \mathbb{N} \)
\[ \frac{r}{s} \otimes \frac{p}{q} = \frac{r}{sq} \cdot q \otimes \frac{p}{q} = \frac{r}{sq} \otimes \frac{pq}{q} = \frac{r}{sq} \otimes p = 0 \]

(b) \( V = \mathbb{C}[x,y]/\langle xy, y - x^2 \rangle \) because of

the following lemma.

Lemma: If \( M \) is an \( R \)-module, \( \mathfrak{a} \subset R \) an ideal
then \( M \otimes _R R/\mathfrak{a} = M/\mathfrak{a}M \).

PF: \( M \otimes _R R/\mathfrak{a} \) is spanned by \( m \otimes 1 \)
and the relations are \( a m \otimes 1 = m \otimes a = 0 \), \( a \in \mathfrak{a} \).

Now, if \( xy = 0 \) and \( y - x^2 = 0 \) then \( x^3 = 0 \),
\( y = x^3 \), so 
\( \mathbb{C}[x,y]/\langle xy, y - x^2 \rangle = \mathbb{C}[x]/\langle x^3 \rangle \), \( \text{dim} V = 3 \)
4. (5 points each question) (a) For which positive integer $n$ does there exist a maximal ideal in $\mathbb{Q}[x]$ of codimension $n$ over $\mathbb{Q}$?

(b) For which positive integer $n$ does there exist a maximal ideal in $\mathbb{R}[x]$ of codimension $n$ over $\mathbb{R}$?

(c) Describe maximal ideals in $\mathbb{R}[x, y]$.

(d) Describe maximal ideals in $\mathbb{Z}[i]$, where $i^2 = -1$. (Hint: show and use that $-1$ is a square modulo a prime $p$ iff $p = 1 \bmod 4$).

4. Any $n$. Can take $m = \langle x^n - 27 \rangle$. This is maximal since $\mathbb{Q}[x]/\langle x^n - 27 \rangle$ is a field.

(b) $n = 1$ or $2$. The quotient must be a finite field extension of $\mathbb{R}$, so it is $\mathbb{R} \cong \mathbb{C}$.

\[ \mathbb{R}[x]/\langle x^2 + 1 \rangle = \mathbb{C}, \quad \mathbb{R}[x]/\langle x^3 \rangle = \mathbb{R}. \]

(c) 1. $\forall x, y \in \mathbb{R}, \quad M = \{ f | f(x_0, y_0) = 0 \}$.

2. A pair of complex conjugate points $(x_0, y_0) \neq (\overline{x_0}, \overline{y_0})$, $M = \{ f | f(x, y) = f(x, \overline{y}) = 0 \}$.

(d) Let $m \in \mathbb{Z}[i]$ be a maximal ideal.

Then $\exists x \in m$, $x \neq 0$ (as $\mathbb{Z}[i]$ is not a field).

Then $0 = x\overline{x} \in m$, $n \in \mathbb{N}$. So, $n = 0$ in $F = \mathbb{Z}[i]/m$.

Hence $F$ is a field of characteristic $p > 0$, and we have a surjection $\mathbb{F}_p[i] \to F$, where $i = \sqrt{-1}$.

If $i \not\in \mathbb{F}_p \implies (p = 1 \bmod 4)$, then $F = \mathbb{F}_p$.

If $i \in \mathbb{F}_p \implies (p = 3 \bmod 4)$, then $F = \mathbb{F}_p$, $p = 2$.

So $M = \langle p, i \rangle$, where $i^2 = -1 \bmod p$. (in fact, this ideal is principal as $\mathbb{Z}[i]$ is a principal ideal).

If $\sqrt{-1} \not\in \mathbb{F}_p \implies (p = 1 \bmod 4)$ and $p \neq 2$ then $M = \langle p \rangle$. (as $\mathbb{Z}[i]$ is a principal ideal but it does not matter).
5. (10 points) Let $R = \mathbb{C}[x, y]$, $R' = \mathbb{C}[[x, y]] \supset R$. Give an example of a prime ideal $p \subset R$ such that $pR' \neq R'$ but $pR'$ is NOT a prime ideal in $R'$.

Take $p = \langle x^2(x+1) - y^2 \rangle$. Since $f$ is irreducible, $p$ is prime. But $f$ splits in $\mathbb{C}[[x, y]]$:

$$f = (x \sqrt{x+1} - y)(x \sqrt{x+1} + y), \text{ where } \sqrt{x+1}$$

is the Taylor series of the function $\sqrt{x+1}$.

Hence, $pR'$ is not a prime ideal in $R'$ (and $pR' \neq R'$).