18,100BC final exam
December 18, 2007

NAME:

1. (20 points) If $S \subset \mathbb{R}$, show that the set of isolated points of S is at most countable (use that the set of rational numbers is countable).

2. Answer the following questions without proofs (each question is 5 points).

Consider the following two subsets of the complex plane: $A = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$, and $B = \{z \in \mathbb{C} \,|\, |z - 1| < 1\}$.

Equip the complex plane with the usual Euclidean metric, and consider the sets

1) A
2) B
3) $A \cap B$
4) $A \cup B$
5) $A \cap B^c$
6) $B \cap A^c$.

Which of the sets 1-6 are

(i) open?

Answer:

(ii) closed?

Answer:

(iii) convex?

Answer:

(iv) compact?

Answer:

Hint. Draw a picture.

3. (5 points each question) Assume that a positive term series

$$\sum_{n=1}^{\infty} a_n$$

is divergent. Determine, with proofs, whether the following series are always convergent, always divergent, or may be either convergent or divergent, depending on the sequence a_n.

(i)

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}.$$

(ii)

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}.$$

(iii)

$$\sum_{n=1}^{\infty} \frac{a_n}{1+n^2a_n}.$$

4. (20 points) Let f be a real uniformly continuous function on a bounded subset E of the real line. Show that f is bounded.

5. Answer the following questions without proof (each question is 5 points).

(i) Find $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{2n^2+n}$.

Answer:

(ii) For which complex x is the series $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ convergent?

Answer:

(iii) Give an example of a continuous unbounded function on $[0,1]\cap\mathbb{Q}$.

Answer:

(iv) Let $a>0$, and f_a be the function on $\mathbb{R}^2$ defined by the formula $f_a(x,y)=\frac{xy}{(x^2+y^2)^a}$ when $(x,y)\neq(0,0)$, and $f_a(0,0)=0$. For which a is f continuous?

Answer:

6. (20 points) Let $f(x)$ be a polynomial with real coefficients whose roots are all real (i.e., f factors as a product of real linear functions). Show that the same property holds for the derivative $f'(x)$.

7. (20 points) Suppose $f(x) \geq 0$ and f is a continuous monotonically decreasing function on $[1, \infty)$. Prove that $\int_1^\infty f(x)dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

8. Answer the following questions without proof (each question is 5 points)

(i) Give an example of a nonnegative continuous function on $[0, \infty)$ such that $\int_0^\infty f(x)dx$ exists (in $\mathbb{R}$), but f is unbounded. Hint: think of what the graph of such f could look like.

Answer:

(ii) Calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{k^2+n^2}$$

Hint: this is an integral sum for an integral.

Answer:

(iii) Compute the Riemann-Stieltjes integral $\int_0^\pi \frac{\sin(x)}{2x} d(x^2)$.

Answer:

(iv) Let $f : [0,1] \to (0,\infty)$ be a differentiable function, such that $f(0) = 1$ and $f(1) = 4$. Find all $\lambda \in \mathbb{R}$ for which there must exist $x \in (0,1)$ such that $f'(x)^2 - \lambda f(x) = 0$. Hint: Apply the mean value theorem to the function $\sqrt{f(x)}$.

Answer:

9. (20 points) Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of $x \in \mathbb{R}$.

10. Answer the following questions without proofs (each question is 5 points).

(i) Give an example of an integrable function on $[0, 1]$ which cannot be the limit of a uniformly convergent sequence of polynomials.

Answer:

(ii) For which $a > 0$ is the sequence of functions $\{x^n, n \geq 1\}$ equicontinuous on $[0, a]$?

Answer:

(iii) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n + 3^n}.$$

Answer:

(iv) Find the limit

$$\lim_{x \to 0} \frac{e^{e^x} - e}{x}.$$

Answer: