## CHAPTER 7

## The Kunita-Watanabe Extension

A careful examination of the results in Section 5.1 and 5.3 reveals that they depend very little on detailed properties of Brownian motion and, in fact, that analogous results can be derived about any square-integrable martingale $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ with the properties that
(1) The $t \rightsquigarrow M(t)$ is $\mathbb{P}$-almost surely continuous.
(2) There is an $\left\{\mathcal{F}_{t}: t \geq 0\right\}$-progressively measurable $A:[0, \infty) \times \Omega$ $\longrightarrow[0, \infty)$ such that $t \rightsquigarrow A(t)$ is $\mathbb{P}$-almost surely continuous and non-decreasing, $A(0)=0$, and $\left(M(t)^{2}-A(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a martingale.

In the case of an $\mathbb{R}$-valued Brownian motion $\left(\beta(t), \mathcal{F}_{t}, \mathbb{P}\right), A(t)=t$. In the case when $t \rightsquigarrow \beta(t)$ is $\mathbb{R}^{n}$-valued and $X(t)=(\xi, \beta(t))_{\mathbb{R}^{n}}$ for some $\xi \in \mathbb{R}^{n}$, $A(t)=t|\xi|^{2}$. More generally, if $\theta \in \Theta^{2}\left(\mathbb{P} ; \mathbb{R}^{n}\right)$ and $M=I_{\theta}$, then $A(t)=$ $\int_{0}^{t}|\theta(\tau)|^{2} d \tau$.

Although J.L. Doob (cf. Chapter 6 of [6]) was the first to recognize that these are the only ingredients which are essential for Itô's theory, it was Kunita and Watanabe [21] who first accomplished the elegant extension of Itô's theory which will we present here. However, before we can do so, we need to have a special, and particularly simple, case of the renowned DoobMeyer Decomposition Theorem for submartingales. ${ }^{1}$

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\left\{\mathcal{F}_{t}: t \geq 0\right\}$ is a non-decreasing family of $\mathbb{P}$-complete sub $\sigma$-algebras of $\mathcal{F}$. Also, when I say that a stochastic process $X$ on $[0, \infty) \times \Omega$ with values in a topological space is $\mathbb{P}$-almost surely right-continuous or continuous, I will mean that $t \rightsquigarrow X(t, \omega)$ is right-continuous or continuous for $\mathbb{P}$-almost every $\omega$.

### 7.1 Doob-Meyer for Continuous Martingales

Recall (cf. Lemma 5.2.18 in [36]) Doob's Decomposition Lemma for discrete parameter, integrable submartingales $\left(X(m), \mathcal{F}_{m}, \mathbb{P}\right)$ : if $A_{0} \equiv 0$ and $A(m)-A(m-1) \equiv \mathbb{E}^{\mathbb{P}}\left[X(m)-X(m-1) \mid \mathcal{F}_{m-1}\right] \vee 0$ for $m \geq 1$, then

[^0]$\{A(m): m \geq 0\}$ is the $\mathbb{P}$-almost surely uniquely determined by the facts that $A(0) \equiv 0, A(m-1)$ is $\mathcal{F}_{m-1}$-measurable for each $m \geq 1$, and $(M(m)-$ $\left.A(m), \mathcal{F}_{m}, \mathbb{P}\right)$ is a martingale. Although, aside from recognizing its potential importance, this lemma requires no effort in the discrete setting, even formulating its generalization to the continuous parameter setting was a major achievement of P.A. Meyer and can be seen as the cornerstone of what became the Strasbourg School of Probability.

Fortunately for us, most of the difficulties Meyer had to overcome disappear when the submartingale is the square of a continuous martingale. Indeed, in this case the program is really an application of the Itô's ideas. In fact, it was Itô who gave me the outline for the existence proof given below.
7.1.1. Uniqueness. Even without delving into the details, it is easy to appreciate the major difficulty confronting Meyer. Namely, when the time parameter is continuous, what plays the role of $\mathcal{F}_{m-1}$ ? That is, what is the measurability property which one has to impose on the process $A$ ? Loosely speaking, Meyer's answer was that $t \rightsquigarrow A(t)$ must be amenable to the reasoning contained in the corollary to the following theorem. What this corollary shows is that continuity is sufficient. One of the key observations made by Meyer is that continuity is not necessary. However, its replacement is subtle. (Cf. Exercise 7.1.4 below.)
7.1.1 Theorem. Suppose that $V:[0, \infty) \times \Omega \longrightarrow \mathbb{R}$ is a progressively function with the properties that, $\mathbb{P}$-almost every $\omega, t \rightsquigarrow V(t, \omega)$ is a rightcontinuous function of locally bounded variation; and use $|V|(t, \omega)$ to denote the variation of $V(\cdot, \omega) \upharpoonright[0, t]$. Then $|V|$ is again progressively measurable. Next, suppose that $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a right-continuous martingale with the property that, for every $t \geq 0$,

$$
\mathbb{E}^{\mathbb{P}}\left[\|M(\cdot)\|_{[0, t]}(|V(0)|+|V|(t))\right]<\infty
$$

Then $\left(M(t) V(t)-B(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a martingale when

$$
B(t, \omega) \equiv \begin{cases}\int_{(0, t]} M(\tau, \omega) V(d \tau, \omega) & \text { if }\|M(\cdot, \omega)\|_{[0, t]}|V|(t, \omega)<\infty \\ 0 & \text { otherwise },\end{cases}
$$

where the $V(d \tau, \omega)$ is meant to be Lebesgue integration with respect to the (signed) measure on $[0, \infty)$ determined by $V(\cdot, \omega)$.
Proof: To see that $|V|$ is progressively measurable, simply observe that, because of right-continuity,

$$
|V|(t, \omega)=\sup _{N \in \mathbb{N}} \sum_{m=0}^{\infty}\left|V\left(t \wedge(m+1) 2^{-N}\right)-V\left(t \wedge m 2^{-N}\right)\right|
$$

Knowing this, it is easy to see that $B$ is progressively measurable and (cf. Exercise 1.2 .29 in [34]) $\mathbb{P}$-almost surely right-continuous. Finally, using the assumed integrability properties, one can easily justify the computation:

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{P}}\left[B\left(t_{2}\right)-B\left(t_{1}\right), A\right] \\
& =\lim _{N \rightarrow \infty} \sum_{m=\left[2^{N} t_{1}\right]}^{\left[2^{N} t_{2}\right]} \mathbb{E}^{\mathbb{P}}\left[M\left(t_{2} \wedge(m+1) 2^{-N}\right)\right. \\
& \left.\quad \times\left(V\left(t_{2} \wedge(m+1) 2^{-N}\right)-V\left(t_{1} \vee m 2^{-N}\right)\right), A\right] \\
& =\lim _{N \rightarrow \infty} \sum_{m=\left[2^{N} t_{1}\right]}^{\left[2^{N} t_{2}\right]} \mathbb{E}^{\mathbb{P}}\left[M\left(t_{2}+1\right)\left(V\left(t_{2} \wedge(m+1) 2^{-N}\right)-V\left(t_{1} \vee m 2^{-N}\right)\right), A\right] \\
& =\mathbb{E}^{\mathbb{P}}\left[M\left(t_{2}+1\right)\left(V\left(t_{2}\right)-V\left(t_{1}\right)\right), A\right]=\mathbb{E}^{\mathbb{P}}\left[M\left(t_{2}\right) V\left(t_{2}\right)-M\left(t_{1}\right) V\left(t_{1}\right), A\right]
\end{aligned}
$$

for all $0 \leq t_{1}<t_{2}$ and $A \in \mathcal{F}_{t_{1}}$.
7.1.2 Corollary. Suppose $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous local martingale, and let $|M|(t, \omega)$ denote the variation of $M(\cdot, \omega) \upharpoonright[0, t]$. Then

$$
\mathbb{P}(\exists \in[0, \infty) 0<|M|(t)<\infty)=0
$$

In particular, if $X:[0, \infty) \times \Omega \longrightarrow \mathbb{R}$ is progressively measurable, then there is, up to a $\mathbb{P}$-null set, at most one progressively measurable $A:[0, \infty) \times \Omega \longrightarrow$ $\mathbb{R}$ with the properties that $t \rightsquigarrow A(t)$ is $\mathbb{P}$-almost surely continuous and of locally bounded variation, $A(0) \equiv 0$, and $\left(X(t)-A(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a local martingale.
Proof: Without loss in generality, assume that $M(0)=0$. Next, given $R>0$, set $\zeta_{R}(\omega)=\sup \{t \geq 0:|M|(t, \omega) \leq R\}$, observe ${ }^{2}$ that $\zeta_{R}$ is a stopping time, and set $M_{R}(t)=M\left(t \wedge \zeta_{R}\right)$. By Doob's Stopping Time Theorem, $\left(M_{R}(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous martingale. At the same time, $\left|M_{R}\right|(t, \omega) \leq R$. Hence, by the preceding theorem,

$$
\left(M_{R}(t)^{2}-\int_{0}^{t} M_{R}(\tau) M_{R}(d \tau), \mathcal{F}_{t}, \mathbb{P}\right)
$$

is a continuous martingale. In particular, this means that

$$
\mathbb{E}^{\mathbb{P}}\left[M\left(t \wedge \zeta_{R}\right)^{2}\right]=\mathbb{E}^{\mathbb{P}}\left[\int_{0}^{t} M_{R}(\tau) M_{R}(d \tau)\right]
$$

[^1]On the other hand, because $M_{R}(\cdot, \omega)$ is continuous, as well as of bounded variation, integration by parts leads to the pathwise identity

$$
M_{R}(t, \omega)^{2}=2 \int_{0}^{t} M_{R}(\tau, \omega) M_{R}(d \tau, \omega) \quad \text { for } \mathbb{P} \text {-almost every } \omega
$$

Hence, after combining this with the above, we conclude that $\mathbb{E}^{\mathbb{P}}\left[M_{R}(t)^{2}\right]=$ 0 . Finally, suppose that $\mathbb{P}(\exists t \geq 00<|M|(t)<\infty)>0$. Then there would exist an $R>0$ and $t \in(0, \infty)$ such that $\mathbb{P}\left(\zeta_{R} \leq t\right)>0$, which would lead to the contradiction $\mathbb{E}^{\mathbb{P}}\left[M_{R}(t)^{2}\right] \geq \frac{1}{4} \mathbb{E}^{\mathbb{P}}\left[\|M\|_{[0, T]}^{2}\right]>0$.

To complete the proof, suppose that $A$ and $A^{\prime}$ are two functions with the described properties. Then $\left(A(t)-A^{\prime}(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous local martingales whose paths are of locally bounded variation. Hence, by what we have just proved, this means that $A=A^{\prime} \mathbb{P}$-almost surely.
7.1.2. Existence. In this subsection, we will show that if $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous, $\mathbb{R}$-valued local martingale, then there exists a $\mathbb{P}$-almost surely unique progressively measurable $A:[0, \infty) \times \Omega \longrightarrow[0, \infty)$ such that $A(0)=$ $0, t \rightsquigarrow A(t)$ is $\mathbb{P}$-almost surely continuous and non-decreasing, and $\left(M(t)^{2}-\right.$ $\left.A(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a local martingale.

To begin, notice that Corollary 7.1.2 provides us with the required uniqueness. Next, observe that it suffices to prove existence in the case when $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a bounded martingale with $M(0) \equiv 0$. Indeed, if this is not already the case, we can take $\zeta_{m}(\omega)=\inf \{t \geq 0:|M(t, \omega)| \geq m\}$ and set $M_{m}(t)=M\left(t \wedge \zeta_{m}\right)$. Assuming that $A_{m}$ exists for each $m \in \mathbb{Z}^{+}$, we would know, by Doob's Stopping Time Theorem and uniqueness, that $A_{m+1} \upharpoonright\left[0, \zeta_{m}\right)=A_{m} \upharpoonright\left[0, \zeta_{m}\right) \mathbb{P}$-almost surely for all $m \geq 1$. Hence, we could construct $A$ by taking $A(t)=\sup \left\{A_{m}(t): m\right.$ with $\left.\zeta_{m} \geq t\right\}$.

Now assume that $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a bounded, continuous martingale with $M(0)=0$. For convenience, we will assume that $M(\cdot, \omega)$ is continuous for every $\omega \in \Omega$. The idea behind Itô's construction of $A$ is to realization that, if $A$ exists, then Itô's formula would hold when $t$ is systematically replaced by $A(t)$. In particular, one would have $M(t)^{2}=2 \int_{0}^{t} M(\tau) d M(\tau)+A(t)$. Thus, it is reasonable to see what happens when we take $A(t) \equiv M(t)^{2}-$ $2 \int_{0}^{t} M(\tau) M(d \tau)$. Of course, this line of reasoning might seem circular since we want $A$ in order to construct stochastic integrals with respect to $M$, but entry into the circle turns out to be easy.

Set $\zeta_{m, 0}(\omega)=m$ for $m \in \mathbb{N}$. Next, proceeding by induction, define $\left\{\zeta_{m, N}\right\}_{m=0}^{\infty}$ for $N \in \mathbb{Z}^{+}$so that $\zeta_{0, N} \equiv 0$ and, for $m \in \mathbb{Z}^{+}, \zeta_{m, N}(\omega)$ is equal to

$$
\begin{gathered}
\zeta_{\ell, N-1}(\omega) \wedge \inf \left\{t \geq \zeta_{m-1, N}(\omega):\left|M(t, \omega)-M\left(\zeta_{m-1, N}(\omega), \omega\right)\right| \geq \frac{1}{N}\right\} \\
\text { for the } \ell \in \mathbb{Z}^{+} \text {with } \zeta_{\ell-1, N-1}(\omega) \leq \zeta_{m-1, N}(\omega)<\zeta_{\ell, N-1}(\omega)
\end{gathered}
$$

For each $N \in \mathbb{N},\left\{\zeta_{m, N}: m \geq 0\right\}$ is a non-decreasing sequence of bounded stopping times which tend to $\infty$ as $m \rightarrow \infty$. Further, these sequences are nested in the sense that $\left\{\zeta_{m, N-1}: m \geq 0\right\} \subseteq\left\{\zeta_{m, N}: m \geq 0\right\}$ for every $N \in \mathbb{Z}^{+}$.

Now set

$$
\begin{aligned}
M_{m, N}(\omega) & =M\left(\zeta_{m, N}(\omega), \omega\right) \quad \text { and } \\
\Delta_{m, N}(t, \omega) & =M\left(t \wedge \zeta_{m, N}(\omega), \omega\right)-M\left(t \wedge \zeta_{m-1, N}(\omega), \omega\right)
\end{aligned}
$$

and observe that

$$
M(t, \omega)^{2}-M(0, \omega)^{2}=2 Y_{N}(t, \omega)+A_{N}(t, \omega)
$$

where

$$
Y_{N}(t, \omega) \equiv \sum_{m=1}^{\infty} M_{m-1, N}(\omega) \Delta_{m, N}(t, \omega) \text { and } A_{N}(t, \omega) \equiv \sum_{m=1}^{\infty} \Delta_{m, N}(t, \omega)^{2}
$$

Furthermore, $\left(Y_{N}(t), \mathcal{F}_{t}, P\right)$ is a continuous martingale, and $A_{N}:[0, \infty) \times$ $\Omega \longrightarrow[0, \infty)$ is a progressively measurable function with the properties that, for each $\omega \in \Omega$ : $A_{N}(0, \omega)=0, A_{N}(\cdot, \omega)$ is a continuous, and $A_{N}(t, \omega)+\frac{1}{N^{2}} \geq$ $A_{N}(s, \omega)$ whenever $0 \leq s<t$. Thus, we will be done if we can prove that, for each $T \in[0, \infty),\left\{\bar{A}_{N}: N \geq 0\right\}$ converges in $L^{2}(\mathbb{P} ; C([0, T] ; \mathbb{R})$, which is equivalent to showing that $\left\{Y_{N}: N \geq 0\right\}$ converges there.

With this in mind, for each $0 \leq N<N^{\prime}$ and $m \in \mathbb{Z}^{+}$, define

$$
M_{m, N^{\prime}}^{(N)}(\omega)=M_{\ell, N}(\omega) \quad \text { when } \zeta_{\ell-1 ; N}(\omega) \leq \zeta_{m, N^{\prime}}(\omega)<\zeta_{\ell, N}(\omega)
$$

and note that

$$
Y_{N^{\prime}}(t, \omega)-Y_{N}(t, \omega)=\sum_{m=1}^{\infty}\left(M_{m, N^{\prime}}(\omega)-M_{m, N^{\prime}}^{(N)}(\omega)\right) \Delta_{m, N^{\prime}}(t, \omega)
$$

Because $\left|M_{m, N^{\prime}}(\omega)-M_{m, N^{\prime}}^{(N)}(\omega)\right| \leq \frac{1}{N}$ and the terms in the series are orthogonal,

$$
\mathbb{E}^{P}\left[\left(Y_{N^{\prime}}(t)-Y_{N}(t)\right)^{2}\right] \leq N^{-2} \mathbb{E}^{P}\left[M(t)^{2}\right]
$$

In particular, as an application of Doob's Inequality, we see first that, for each $T \geq 0$,

$$
\lim _{N \rightarrow \infty} \sup _{N^{\prime}>N} \mathbb{E}^{\mathbb{P}}\left[\left\|Y_{N^{\prime}}-Y_{N}\right\|_{[0, T]}^{2}\right]=0
$$

and then that there exists a continuous martingale $\left(Y(t), \mathcal{F}_{t}, \mathbb{P}\right)$ with the property that

$$
\lim _{N \rightarrow \infty} \mathbb{E}^{P}\left[\left\|Y_{N}-Y\right\|_{[0, T]}^{2}\right]=0 \quad \text { for each } T \in[0, \infty)
$$

To complete the proof at this point, define the function $A:[0, \infty) \times \Omega \longmapsto$ $[0, \infty)$ so that

$$
A(t, \omega)=0 \vee \sup \left\{M(s, \omega)^{2}-2 Y(s, \omega): s \in[0, t]\right\}
$$

and check that $A$ has the required properties. Hence, we have now proved the following version of the Doob-Meyer Decomposition Theorem.
7.1.3 Theorem. If $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous, $\mathbb{R}$-valued local martingale, then there exists a $\mathbb{P}$-almost surely unique progressively measurable function $A:[0, \infty) \times \Omega \longrightarrow[0, \infty)$ with the properties that $A(0) \equiv 0$, $t \rightsquigarrow A(t)$ is $\mathbb{P}$-almost surely continuous and non-decreasing, and $\left(M(t)^{2}-\right.$ $\left.A(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a local martingale.

From now on, we will use the notation $\langle M\rangle$ to denote the process $A$ described in Theorem 7.1.3.

### 7.1.3. Exercises.

EXERCISE 7.1.4. Because we have not considered martingales with discontinuities, the most subtle aspects of Meyer's Theorem are not apparent in our treatment. To get a feeling for what these subtleties are, consider a simple Poisson process (cf. §1.4.2) $N(t)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{F}_{t}$ be the $\mathbb{P}$-completion of $\sigma(N(\tau): \tau \in[0, t])$, set $M(t)=N(t)-t$, and check that $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a non-constant martingale. At the same time, $t \rightsquigarrow M(t) \mathbb{P}$-almost surely has locally bounded variation. Hence, the first part of Corollary 7.1.2 is, in general, false unless one imposes some condition on the paths $t \rightsquigarrow M(t)$. The condition which we imposed was continuity. However, a look at the proof reveals that the only place where we used continuity was when we integrated by parts to get $M_{R}(t)=2 \int_{0}^{t} M_{R}(\tau) M_{R}(d \tau)$. This is the point alluded to in the rather cryptic remark preceding Theorem 7.1.1.

ExERCISE 7.1.5. Let $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ be a continuous local martingale, $\zeta$ a stopping time, and set $M^{\zeta}(t)=M(t \wedge \zeta)$.
(i) Show that $\left\langle M^{\zeta}\right\rangle(t)=\langle M\rangle(t \wedge \zeta)$.
(ii) If $\langle M\rangle(\zeta) \in L^{1}(\mathbb{P} ; \mathbb{R})$, show that

$$
\left(M^{\zeta}(t)-M(0), \mathcal{F}_{t}, \mathbb{P}\right) \quad \text { and } \quad\left(\left(M^{\zeta}(t)-M(0)\right)^{2}-\left\langle M^{\zeta}\right\rangle(t), \mathcal{F}_{t}, \mathbb{P}\right)
$$

are martingales.
(iii) Suppose $\alpha: \Omega \longrightarrow \mathbb{R}$ is an $\mathcal{F}_{\zeta}$-measurable function, and set $M^{\prime}(t)=$ $\alpha(M(t)-M(t \wedge \zeta))$. After checking that $\left(M^{\prime}(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous local martingale, show that

$$
\left\langle M^{\prime}\right\rangle(t)=\alpha^{2}(\langle M\rangle(t)-\langle M\rangle(t \wedge \zeta))
$$

ExErcise 7.1.6. Let $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ be a continuous, $\mathbb{R}$-valued local martingale.
(i) Show that $M(\infty) \equiv \lim _{t \rightarrow \infty} M(t)$ exists $\mathbb{P}$-almost surely on the set $\{\langle M\rangle(\infty)<\infty\}$.
Hint: Let $\zeta_{R}=\inf \{t \geq 0:\langle M\rangle \geq R\}$, show that $\left(M\left(t \wedge \zeta_{R}\right)-M(0), \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous martingale whose second moment is bounded by $R$, and apply the Martingale Convergence Theorem (cf. Theorem 7.1.16 in [36]) to conclude that $\lim _{t \rightarrow \infty} M\left(t \wedge \zeta_{R}\right)$ exists $\mathbb{P}$-almost surely.
(ii) Let $\zeta$ be a stopping time with the property that $\langle M\rangle(\zeta)<\infty \mathbb{P}$-almost surely, and, using part (i), define $M(\zeta)$ on $\{\zeta=\infty\}$ equal to be $\mathbb{P}$-almost surely equal to $\lim _{t / \infty} M(t)$. Show that $\langle M\rangle(\zeta) \in L^{1}(\mathbb{P} ; \mathbb{R})$ if and only if $M(\zeta) \in L^{2}(\mathbb{P} ; \mathbb{R})$, in which case $\left.\mathbb{E}^{\mathbb{P}}[(M(\zeta)-M 90))^{2}\right]=\mathbb{E}^{\mathbb{P}}[\langle M\rangle(\zeta)]$.
ExERCISE 7.1.7. Suppose that $\left\{M_{k}\right\}_{1}^{\infty} \subseteq \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ and that $\zeta$ is a stopping time. If $\left\langle M_{k}\right\rangle(\zeta) \longrightarrow 0$ in $\mathbb{P}$-probability, show that $\left\|M_{k}-M_{k}(0)\right\|_{[0, \zeta)} \longrightarrow \square$ 0 in $\mathbb{P}$-probability.

ExERCISE 7.1.8. Let a continuous, $\mathbb{R}$-valued, local martingale $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ be given, and, for each $\omega$, use $G(\omega)$ to denote the set of all $t \in(0, \infty)$ for which there exist $0 \leq a<t<b<\infty$ with $\langle M\rangle(b, \omega)=\langle M\rangle(a, \omega)$. Clearly, $G(\omega)$ is an open subset of $\mathbb{R}$, and as such its connected components are open intervals. The goal of this exercise is to show that, for $\mathbb{P}$-almost every $\omega$, $M(\cdot, \omega)$ is constant on each connected component of $G(\omega)$.
(i) For each $t \in[0, \infty)$, define $\sigma(t, \omega)=\sup \{\tau \geq t:\langle M\rangle(\tau, \omega)=$ $\langle M\rangle(t, \omega)\}$. Show that $M(\cdot, \omega)$ is constant on each connected component of $G(\omega)$ if and only if $M(t, \omega)=M(\sigma(t, \omega), \omega)$ for each rational number $t \in[0, \infty)$.
(ii) In view of (i) and the $\mathbb{P}$-almost sure continuity of $M(\cdot, \omega)$, we will have reached our goal once we show that, for each $0 \leq t<\tau<\infty, M(\tau \wedge$ $\sigma(t, \omega), \omega)=M(t, \omega) \mathbb{P}$-almost surely. Prove this first in the case when $M$ is a square-integrable martingale, and then reduce to this case by a stopping time argument.

### 7.2 Kunita-Watanabe Stochastic Integration

Recall (cf. §5.1.3) the notation $\mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ for the space of all $\mathbb{R}$-valued, continuous local martingales on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$
relative to the non-decreasing family $\left\{\mathcal{F}_{t}: t \geq 0\right\}$ of $\mathbb{P}$-complete sub $\sigma$ algebras. Given an $M \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$, let $\Theta_{\text {loc }}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$ denote the space of progressively measurable $\theta:[0, \infty) \longrightarrow \mathbb{R}$ with the property that

$$
\int_{0}^{T}|\theta(\tau)|^{2}\langle M\rangle(d \tau)<\infty \quad \mathbb{P} \text {-almost surely for all } T \in[0, \infty)
$$

Following Kunita and Watanabe, we will define in this section the stochastic integral $I_{\theta}^{M} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ of $\theta \in \Theta_{\mathrm{loc}}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$ with respect to $M \in$ $\mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$.
7.2.1. The Hilbert Structure of $\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$. Clearly $\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ is a vector space and $M \rightsquigarrow\langle M\rangle$ is some sort of non-negative, quadratic functional on this vector space. In particular, these trivial observations, in conjuction with Corollary 7.1.2, lead immediately to the conclusion that for each pair $\left(M_{1}, M_{2}\right) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})^{2}$ there is a $\mathbb{P}$-almost surely unique progressively measurable $\left\langle M_{1}, M_{2}\right\rangle:[0, \infty) \times \Omega \longrightarrow \mathbb{R}$ with the properties that $\left\langle M_{1}, M_{2}\right\rangle(0)=0, t \rightsquigarrow\left\langle M_{1}, M_{2}\right\rangle(t)$ is $\mathbb{P}$-almost surely continuous and of locally bounded variation, and

$$
\left(M_{1}(t) M_{2}(t)-\left\langle M_{1}, M_{2}\right\rangle(t), \mathcal{F}_{t}, \mathbb{P}\right) \quad \text { is a local martingale. }
$$

Indeed, the uniqueness is immediate from Corollary 7.1.2 and the existence is an application of polarization: ${ }^{3}$

$$
\left\langle M_{1}, M_{2}\right\rangle=\frac{\left\langle M_{1}+M_{2}\right\rangle-\left\langle M_{1}-M_{2}\right\rangle}{4} .
$$

7.2.1 Theorem. The map $\left(M_{1}, M_{2}\right) \rightsquigarrow\left\langle M_{1}, M_{2}\right\rangle$ is, symmetric, bilinear, and non-negative in the sense that, $\mathbb{P}$-almost surely: $\left\langle M_{1}, M_{2}\right\rangle=\left\langle M_{2}, M_{1}\right\rangle$, $\left\langle\alpha_{1} M_{1}+\alpha_{2} M_{2}, M_{3}\right\rangle=\alpha_{1}\left\langle M_{1}, M_{3}\right\rangle+\alpha_{2}\left\langle M_{2}, M_{3}\right\rangle$, and $\langle M, M\rangle \geq 0$. Moreover,

$$
\begin{align*}
\mid\left\langle M_{1}, M_{2}\right\rangle\left(t_{2}\right) & -\left\langle M_{1}, M_{2}\right\rangle\left(t_{1}\right) \mid \\
& \leq \sqrt{\left\langle M_{1}\right\rangle\left(t_{2}\right)-\left\langle M_{1}\right\rangle\left(t_{1}\right)} \sqrt{\left\langle M_{2}\right\rangle\left(t_{2}\right)-\left\langle M_{2}\right\rangle\left(t_{1}\right)}  \tag{7.2.2}\\
\text { for all } 0 & \leq t_{1}<t_{2} \mathbb{P} \text {-almost surely. }
\end{align*}
$$

Equivalently, $\left\langle M_{1}, M_{2}\right\rangle$ is $\mathbb{P}$-almost surely absolutely continuous with respect to $\mu_{\omega} \equiv\left\langle M_{1}\right\rangle(\cdot, \omega)+\left\langle M_{2}\right\rangle(\cdot, \omega)$, and if $f_{i, j}(\cdot, \omega)$ denotes the RadonNikodym derivative of $\left\langle M_{i}, M_{j}\right\rangle(\cdot, \omega)$ with respect to $\mu_{\omega}$, then, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$
\left|f_{1,2}(\cdot, \omega)\right| \leq \sqrt{f_{1,1}(\cdot, \omega) f_{2,2}(\cdot, \omega)} \quad \mu_{\omega} \text {-almost everywhere. }
$$

[^2]In particular, $\mathbb{P}$-almost surely,

$$
\left\|\left\langle M_{2}\right\rangle^{\frac{1}{2}}-\left\langle M_{1}\right\rangle^{\frac{1}{2}}\right\|_{[0, T]} \leq\left\langle M_{2}-M_{1}\right\rangle(T) \quad \text { for all } T \geq 0
$$

Proof: The first assertion requiring comment is the inequality in (7.2.2). To prove it, first note that is suffices to show that for each $0 \leq t_{1}<t_{2}$ and $\alpha>0$,

$$
\begin{align*}
2 \mid\left\langle M_{1}, M_{2}\right\rangle\left(t_{2}\right) & -\left\langle M_{1}, M_{2}\right\rangle\left(t_{1}\right) \mid \\
& \leq \alpha\left(\left\langle M_{1}\right\rangle\left(t_{2}\right)-\left\langle M_{1}\right\rangle\left(t_{1}\right)\right)+\frac{1}{\alpha}\left(\left\langle M_{2}\right\rangle\left(t_{2}\right)-\left\langle M_{2}\right\rangle\left(t_{1}\right)\right) \tag{}
\end{align*}
$$

$\mathbb{P}$-almost surely. Indeed, given $\left(^{*}\right)$, one can easily argue that, $\mathbb{P}$-almost surely, the same inequality holds simultaneously for all $\alpha>0$ and $0 \leq t_{1}<t_{2}$; and once this is known, (7.2.2) follows by the usual minimization procedure with which one proves Schwartz's inequality. But $\left({ }^{*}\right)$ is a trivial consequence of non-negative bilinearity. Namely, for any $\alpha>0$,

$$
\begin{aligned}
& 0 \leq\left\langle\alpha^{\frac{1}{2}} M_{1} \pm \alpha^{-\frac{1}{2}} M_{2}, \alpha^{\frac{1}{2}} M_{1} \pm \alpha^{-\frac{1}{2}} M_{2}\right\rangle\left(t_{2}\right) \\
& \quad-\left\langle\alpha^{\frac{1}{2}} M_{1} \pm \alpha^{-\frac{1}{2}} M_{2}, \alpha^{\frac{1}{2}} M_{1} \pm \alpha^{-\frac{1}{2}} M_{2}\right\rangle\left(t_{1}\right) \\
&=\alpha\left(\left\langle M_{1}\right\rangle\left(t_{2}\right)-\left\langle M_{1}\right\rangle\left(t_{1}\right)\right) \pm 2\left(\left\langle M_{1}, M_{2}\right\rangle\left(t_{2}\right)-\left\langle M_{1}, M_{2}\right\rangle\left(t_{2}\right)\right) \\
&+\alpha^{-1}\left(\left\langle M_{2}\right\rangle\left(t_{2}\right)-\left\langle M_{2}\right\rangle\left(t_{1}\right)\right)
\end{aligned}
$$

$\mathbb{P}$-almost surely.
Knowing the Schwarz inequality for $\left\langle M_{1}, M_{2}\right\rangle$, the triangle inequality

$$
\left|\sqrt{\left\langle M_{2}\right\rangle(t)}-\sqrt{\left\langle M_{1}\right\rangle(t)}\right| \leq\left\langle M_{2}-M_{1}\right\rangle(t) \leq\left\langle M_{2}-M_{1}\right\rangle(T) \quad, 0 \leq t \leq T
$$

$\mathbb{P}$-almost surely follows immediately. Hence, completing the proof from here comes down to showing that if $\mu_{1}$ and $\mu_{2}$ are finite, non-negative, non-atomic Borel measures on $[0, T]$ and $\nu$ is a signed Borel measure on $[0, T]$ satisfying $|\nu(I)| \leq \sqrt{\mu_{1}(I) \mu_{2}(I)}$ for all half-open intervals $I=[a, b) \subseteq[0, T]$, then $\nu \ll \mu \equiv \mu_{1}+\mu_{2}$ and $|g| \leq \sqrt{f_{1} f_{2}} \mu$-almost everywhere, where $g=\frac{d \nu}{d \mu}$ and $f_{i}=\frac{d \mu_{i}}{d \mu}$. Because, for all $\alpha>0,2 \sqrt{\mu_{1}(I) \mu_{2}(I)} \leq \alpha \mu_{1}(I)+\alpha^{-1} \mu_{2}(I)$, the absolute continuity statement is clear. In addition, we have

$$
2\left|\int_{0}^{T} \varphi g d \mu\right| \leq \alpha \int_{0}^{T}|\varphi| f_{1} d \mu+\alpha^{-1} \int_{0}^{T}|\varphi| f_{2} d \mu
$$

first for $\varphi$ 's which are indicator functions of intervals $[a, b)$, next for linear combinations of such functions, then for continuous $\varphi$ 's, and finally for all Borel bounded measurable $\varphi$ 's. But this means that, $\mu$-almost everywhere, $2|g| \leq \alpha f_{1}+\alpha^{-1} f_{2}$ for all $\alpha>0$, and therefore that $|g| \leq \sqrt{f_{1} f_{2}}$.

In the following, and throughout, we will say that a sequence $\left\{M_{k}\right\}_{1}^{\infty}$ in $\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ converges in $\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ to $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ if, for each $T \geq 0$, $\left\langle M_{k}-M\right\rangle(T) \longrightarrow 0$ in $\mathbb{P}$-probability.
7.2.3 Corollary. If $M_{k} \longrightarrow M$ in $\mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$, then

$$
\left\|\left(M_{k}-M_{k}(0)\right)-(M-M(0))\right\|_{[0, T]} \vee\left\|\left\langle M_{k}\right\rangle-\langle M\rangle\right\|_{[0, T]} \longrightarrow 0
$$

in $\mathbb{P}$-probability for each $T \geq 0$.
Moreover, if $\left\{M_{k}\right\}_{1}^{\infty} \subseteq \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ and

$$
\lim _{k \rightarrow \infty} \sup _{\ell \geq k}\left\langle M_{\ell}-M_{k}\right\rangle(T)=0 \quad \text { in } \mathbb{P} \text {-probability for each } T \geq 0
$$

then there exists a $M \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ to which $\left\{M_{k}-M(0)\right\}_{1}^{\infty}$ converges to $M$ in $\mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$.
Proof: Without loss in generality, we will assume that $M_{k}(0)=0=M(0)$ for $k \geq 1$.
In view of the triangle inequality proved in Theorem 7.2 .1 , the only part of the first assertion which requires comment is the proof that $\| M_{k}-$ $M \|_{[0, T]} \longrightarrow 0$ in $\mathbb{P}$-probability for all $T \geq 0$. However, if

$$
\zeta_{R} \equiv \inf \left\{t \geq 0: \sup _{k \geq 1}\left\langle M_{k}\right\rangle(t) \geq R\right\}
$$

then $\zeta_{R} \nearrow \infty \mathbb{P}$-almost surely as $R \rightarrow \infty$ and, by Exercise 7.1.5 and Doob's Inequality,

$$
\mathbb{E}^{\mathbb{P}}\left[\left\|M_{k}-M\right\|_{\left[0, T \wedge \zeta_{R}\right]}^{2}\right] \leq 4 \mathbb{E}^{\mathbb{P}}\left[\left\langle M_{k}-M\right\rangle\left(T \wedge \zeta_{R}\right)\right] \longrightarrow 0
$$

as $k \rightarrow \infty$ for each $R>0$.
Turning to the Cauchy criterion in the second assertion, define $\zeta_{R}$ as in the preceding paragraph, and observe that the argument given there also shows that

$$
\lim _{k \rightarrow \infty} \sup _{\ell \geq k} \mathbb{E}^{\mathbb{P}}\left[\left\|M_{\ell}-M_{k}\right\|_{\left[0, T \wedge \zeta_{R}\right]}^{2}\right]=0
$$

for each $R>0$. Hence, there exists an $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ such that $\| M_{k}-$ $M \|_{[0, T]} \longrightarrow 0$ in $\mathbb{P}$-probability for all $T>0$. At the same time, we know that, for each $R>0$ and $T>0$,

$$
\mathbb{E}^{\mathbb{P}}\left[\left\langle M_{k}-M\right\rangle\left(T \wedge \zeta_{R}\right)\right]=\mathbb{E}^{\mathbb{P}}\left[\left|\left(M_{k}-M\right)\left(T \wedge \zeta_{R}\right)\right|^{2}\right] \longrightarrow 0
$$

as $k \rightarrow \infty$. Hence, for each $T>0,\left\langle M_{k}-M\right\rangle(T) \longrightarrow 0$ in $\mathbb{P}$-probability.
7.2.2. The Kunita-Watanabe Stochastic Integral. The idea of Kunita and Watanabe is to base the definition of stochastic integration on the Hilbert structure described in the preceding subsection. Namely, given
$\theta \in \Theta_{\mathrm{loc}}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$, they say that $I_{\theta}^{M}$ should be the element of $\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ with the properties that

$$
\begin{align*}
& I_{\theta}^{M}(0)=0 \text { and }\left\langle I_{\theta}^{M}, M^{\prime}\right\rangle(t)=\int_{0}^{t} \theta(\tau)\left\langle M, M^{\prime}\right\rangle(d \tau)  \tag{7.2.4}\\
& \text { for all } M^{\prime} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R}) .
\end{align*}
$$

Before adopting this definition, one must check that (7.2.4) makes sense and that, up to a $\mathbb{P}$-null set, it determines a unique element of $\mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$. To handle the first of these, observe that, by Theorem 7.2.1,

$$
\int_{0}^{T}|\theta(\tau)|\left|\left\langle M, M^{\prime}\right\rangle\right|(d \tau) \leq \sqrt{\int_{0}^{T}|\theta(\tau)|^{2}\langle M\rangle(d \tau)\left\langle M^{\prime}\right\rangle(T)}
$$

Hence, $\theta \in \Theta_{\text {loc }}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$ implies that, $\mathbb{P}$-almost surely, $\theta$ is locally integrable with respect to the signed measure $\left\langle M, M^{\prime}\right\rangle$. As for the uniqueness question, suppose that $I$ and $J$ both satisfy (7.2.4), and set $\Delta=I-J$. Then $\langle\Delta\rangle \equiv 0$, and so there exists a non-decreasing sequence $\left\{\zeta_{m}\right\}_{1}^{\infty}$ of stopping times such that $\zeta_{m} \nearrow \infty$ and $\left(\Delta\left(\cdot \wedge \zeta_{m}\right)^{2}, \mathcal{F}_{t}, \mathbb{P}\right)$ is a bounded martingale for each $m$, which, since $\Delta(0) \equiv 0$, means that $\mathbb{E}^{\mathbb{P}}\left[\Delta\left(t \wedge \zeta_{m}\right)^{2}\right]=0$ for all $m \geq 1$ and $t \geq 0$.

Having verified that (7.2.4) makes sense and uniquely determines $I_{\theta}^{M}$, what remains is for us to prove that $I_{\theta}^{M}$ always exists, and, as should come as no surprise, this requires us to return (cf. §5.1.2) to Itô 's technique for constructing his integral. Namely, given $M \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ and a bounded, progressively measurable $\theta: \Omega \longrightarrow \mathbb{R}$ with the property that $\theta(t)=\theta\left([t]_{N}\right)$ for some $N \in \mathbb{N}$, set

$$
I_{\theta}^{M}(t)=\sum_{m=0}^{\infty} \theta\left(m 2^{-N}\right)\left(M\left(t \wedge(m+1) 2^{-N}\right)-M\left(t \wedge m 2^{-N}\right)\right)
$$

Clearly (cf. part (ii) of Exercise 7.1.5), if $\zeta$ is a stopping time for which $\langle M\rangle(\zeta) \in L^{1}(\mathbb{P} ; \mathbb{R})$, then $I_{\theta}^{M}(t \wedge \zeta)$ is $\mathbb{P}$-square integrable for all $t \geq 0$ and, for all $m \in \mathbb{N}$ and $m 2^{-N} \leq t_{1}<t_{2} \leq(m+1) 2^{-N}$,

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{P}}\left[I_{\theta}^{M}\left(t_{2} \wedge \zeta\right)-I_{\theta}^{M}\left(t_{1} \wedge \zeta\right) \mid \mathcal{F}_{t_{1}}\right] \\
& \quad=\theta\left(m 2^{-N}\right) \mathbb{E}^{\mathbb{P}}\left[M\left(t_{2} \wedge \zeta\right)-M\left(t_{1} \wedge \zeta\right) \mid \mathcal{F}_{t_{1}}\right]=0
\end{aligned}
$$

Thus $I_{\theta}^{M} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$. In addition, if $M^{\prime} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ and $\left\langle M^{\prime}\right\rangle(\zeta)$ is also $\mathbb{P}$-integrable, then

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}} & {\left[I_{\theta}^{M}\left(t_{2} \wedge \zeta\right) M^{\prime}\left(t_{2} \wedge \zeta\right)-I_{\theta}^{M}\left(t_{1} \wedge \zeta\right) M^{\prime}\left(t_{1} \wedge \zeta\right) \mid \mathcal{F}_{t_{1}}\right] } \\
& =\theta\left(m 2^{-N}\right) \mathbb{E}^{\mathbb{P}}\left[M\left(t_{2} \wedge \zeta\right) M^{\prime}\left(t_{2} \wedge \zeta\right)-M\left(t_{1} \wedge \zeta\right) M^{\prime}\left(t_{1} \wedge \zeta\right) \mid \mathcal{F}_{t_{1}}\right] \\
& =\theta\left(m 2^{-N}\right) \mathbb{E}^{\mathbb{P}}\left[\left\langle M, M^{\prime}\right\rangle\left(t_{2} \wedge \zeta\right)-\left\langle M, M^{\prime}\right\rangle\left(t_{1} \wedge \zeta\right) \mid \mathcal{F}_{t_{1}}\right] \\
& =\mathbb{E}^{\mathbb{P}}\left[\int_{t_{1} \wedge \zeta}^{t_{2} \wedge \zeta} \theta(\tau)\left\langle M, M^{\prime}\right\rangle(d \tau) \mid \mathcal{F}_{t_{1}}\right],
\end{aligned}
$$

which proves that $\left\langle I_{\theta}^{M}, M^{\prime}\right\rangle(d t)=\theta(t)\left\langle M, M^{\prime}\right\rangle(d t)$. Hence, we now know that $I_{\theta}^{M}$ exists for all bounded, progressively measurable $\theta:[0, \infty) \times \Omega \longrightarrow \mathbb{R}$ with the property that $\theta(t)=\theta\left([t]_{N}\right)$ for some $N \in \mathbb{N}$. Furthermore, by Corollary 7.2.3, we know that if $\left\{\theta_{N}\right\}_{1}^{\infty} \cup\{\theta\} \subseteq \Theta_{\text {loc }}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$ and

$$
\begin{equation*}
\int_{0}^{T}\left|\theta_{N}(\tau)-\theta(\tau)\right|^{2}\langle M\rangle(d \tau) \longrightarrow 0 \quad \text { in } \mathbb{P} \text {-probability for all } T>0 \tag{7.2.5}
\end{equation*}
$$

then $I_{\theta}^{M}$ exists. Hence, we will be done once we prove the following lemma.
7.2.6 Lemma. For each $\theta \in \Theta_{\text {loc }}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$ there exists a sequence $\left\{\theta_{N}\right\}_{1}^{\infty}$ of bounded, $\mathbb{R}$-valued, progressively measurable functions such that $\theta_{N}(t)=$ $\theta_{N}\left([t]_{N}\right)$ and (7.2.5) holds.

Proof: Clearly, it suffices to handle $\theta$ 's which are bounded and vanish off of $[0, T]$ for some $T>0$. In addition, we may assume that $t \rightsquigarrow\langle M\rangle(t, \omega)$ is bounded, continuous and non-decreasing for each $\omega$. Thus, we will make these assumptions.

There is no problem if $t \rightsquigarrow \theta(t, \omega)$ is continuous for all $\omega \in \Omega$, since we can then take $\theta_{N}(t)=\theta\left([t]_{N}\right)$. Hence, what must be shown is that for each bounded, progressively measurable $\theta$ there exists a sequence $\left\{\theta_{N}\right\}_{1}^{\infty}$ of bounded, progressively, $\mathbb{R}$-valued functions with the properties that $t \rightsquigarrow$ $\theta(t, \omega)$ is continuous for each $\omega$ and (7.2.5) holds. To this end, set $A(t, \omega)=$ $t+\langle M\rangle(t, \omega)$, and, for each $s \in[0, \infty)$, determine $\omega \in \Omega \longmapsto \zeta(s, \omega) \in$ $[0, \infty)$ so that $A(\zeta(s, \omega), \omega)=s$. Clearly, for each $\omega, t \rightsquigarrow A(t, \omega)$ is a homeomorphism from $[0, \infty)$ onto itself, and so, an elementary change of variables yields

$$
\begin{equation*}
\int_{[0, \infty)} f(t) A(d t, \omega)=\int_{[0, \infty)} f \circ \zeta(s, \omega) d s \tag{}
\end{equation*}
$$

for any non-negative, Borel measurable $f$ on $[0, \infty)$..
To take the next step, notice that, for each $s, \omega \rightsquigarrow \zeta(s, \omega)$ is a stopping time, and set $\mathcal{F}_{s}^{\prime}$ equal to the $\mathbb{P}$-completion of $\mathcal{F}_{\zeta(s)}$. Thus, if $\theta^{\prime}(s, \omega) \equiv$ $\theta(\zeta(s, \omega), \omega)$, then $\theta^{\prime}:[0, \infty) \times \Omega \longrightarrow \mathbb{R}$ is a bounded function which vanishes off of $[0, A(T)] \times \Omega$ and is progressively measurable with respect to the filtration $\left\{\mathcal{F}_{s}^{\prime}: s \geq 0\right\}$. Hence, by the argument given to prove the density statement in Lemma 5.1.8, we can find a sequence $\left\{\theta_{N}^{\prime}\right\}_{1}^{\infty}$ of $\left\{\mathcal{F}_{s}^{\prime}: s \geq 0\right\}$ progressively measurable such that $t \rightsquigarrow \theta_{N}^{\prime}(t, \omega)$ is bounded, continuous, and supported on $[0, A(1+T, \omega)]$ for each $\omega$, and

$$
\lim _{N \rightarrow \infty} \mathbb{E}^{\mathbb{P}}\left[\int_{0}^{A(1+T)}\left|\theta_{N}^{\prime}(s)-\theta^{\prime}(s)\right|^{2} d s\right]=0
$$

Finally, set $\theta_{N}(t, \omega)=\theta_{N}^{\prime}(A(t, \omega), \omega)$, note that each $\theta_{N}$ is a bounded, $\left\{\mathcal{F}_{t}: t \geq 0\right\}$-progressively measurable function with the properties that $t \rightsquigarrow \theta_{N}(t, \omega)$ is continuous and vanishes off $[0,1+T] \times \Omega$ for each $\omega$. Further, by (*)

$$
\lim _{N \rightarrow \infty} \mathbb{E}^{\mathbb{P}}\left[\int_{0}^{1+T}\left|\theta_{N}(t)-\theta(t)\right|^{2} A(d t)\right]=0
$$

Hence, (7.2.5) holds.
Summarizing the results proved in this subsection, we state the following theorem.
7.2.7 Theorem. For each $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ there is a linear map $\theta \in$ $\Theta_{\mathrm{loc}}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R}) \longmapsto I_{\theta}^{M} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ such that (7.2.4) holds.

Just as we did in the case treated in Chapter 5, we will use the notation $\int_{0}^{t} \theta(\tau) d M(\tau)$ interchangeably with $I_{\theta}^{M}(t)$. More generally, given stopping times $\zeta_{1} \leq \zeta_{2}$, we define

$$
\int_{t \wedge \zeta_{1}}^{t \wedge \zeta_{2}} \theta(\tau) d M(\tau)=I_{\theta}^{M}\left(t \wedge \zeta_{2}\right)-I_{\theta}^{M}\left(t \wedge \zeta_{1}\right)
$$

Starting from Exercise 7.1.5, it is an easy matter to check that

$$
\begin{equation*}
\int_{t \wedge \zeta_{1}}^{t \wedge \zeta_{2}} \theta(\tau) d M(\tau)=\int_{0}^{t} \mathbf{1}_{\left[\zeta_{1}, \zeta_{2}\right)}(\tau) \theta(\tau) d M(\tau) \tag{7.2.8}
\end{equation*}
$$

7.2.3. General Itô 's Formula. Because our proof of Itô's formula in $\S 5.3$ was modeled on the argument given by Kunita and Watanabe, its adaptation to their stochastic integral defined in $\S 7.2 .3$ requires no substantive changes. Indeed, because, by part (i) of Exercise 7.1.5, we already know that, for bounded stopping times $\zeta_{1} \leq \zeta_{2}$

$$
\int_{\zeta_{1}}^{\zeta_{2}} \theta(\tau) d M(\tau)=\int_{0}^{\infty} \mathbf{1}_{[0, \zeta)}(\tau) \theta(\tau) d M(\tau)
$$

the same argument as we used to prove Theorem 5.3.1 allows us to prove the following extension.
7.2.9 Theorem. Let $X=\left(X_{1}, \ldots, X_{k}\right):[0, \infty) \times \Omega \longrightarrow \mathbb{R}^{k}$ and $Y=$ $\left(Y_{1}, \ldots, Y_{\ell}\right):[0, \infty) \times \Omega \longrightarrow \mathbb{R}^{\ell}$ be progressively measurable maps with the properties that, for each $1 \leq i \leq k$ and $\omega, t \rightsquigarrow X_{i}(t, \omega)$ is continuous and of locally bounded variation and, for each $1 \leq j \leq \ell, Y_{j} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$; and set
$Z=(X, Y)$. Then, for each $F \in C^{1,2}\left(\mathbb{R}^{k} \times \mathbb{R}^{\ell} ; \mathbb{R}\right)$,

$$
\begin{aligned}
& F(Z(t))-F(Z(0)) \\
& =\sum_{i=1}^{k} \int_{0}^{t} \partial_{x_{i}} F(Z(\tau)) d X_{i}(\tau)+\sum_{j=1}^{\ell} \int_{0}^{t} \partial_{y_{j}} F(Z(\tau)) d Y_{j}(\tau) \\
& \quad+\frac{1}{2} \sum_{j, j^{\prime}=1}^{\ell} \int_{0}^{t} \partial_{y_{j}} \partial_{y_{j^{\prime}}} F(Z(\tau))\left\langle Y_{j}, Y_{j^{\prime}}\right\rangle(d \tau), \quad t \geq 0
\end{aligned}
$$

$\mathbb{P}$-almost surely. Here, the $d X_{i}$-integrals are taken in the sense of RiemannStieltjes and the $d Y_{j}$-integrals are taken in the sense of Itô, as described in Theorem 7.2.7.

We will again refer to this extension as Itô's formula, and, not surprisingly, there are myriad applications of it. For example, as Kunita and Watanabe pointed out, it gives an elegant proof of the following famous theorem of Paul Lévy.
7.2.10 Corollary. Suppose that $\left\{M_{j}: 1 \leq j \leq n\right\} \subseteq \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ and that $M_{j}(0)=0$ for each $1 \leq j \leq n$. If $\beta=\left(M_{1}, \ldots, M_{n}\right)$, then $\left(\beta(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is an $\mathbb{R}^{n}$-valued Brownian motion if and only if $\left\langle M_{j}, M_{j^{\prime}}\right\rangle(t)=t \delta_{j, j^{\prime}}$.
Proof: We need only discuss the sufficiency. Given $\xi \in \mathbb{R}^{n}$, set

$$
F_{\xi}(t, y)=\exp \left(\sqrt{-1}(\xi, y)_{\mathbb{R}^{n}}+\frac{t}{2}|\xi|^{2}\right)
$$

apply Itô's formula to see that

$$
F_{\xi}(t, M(t))=1+\sqrt{-1} \sum_{j=1}^{n} \int_{0}^{t} \xi_{j} F_{\xi}(\tau, M(\tau)) d M_{j}(\tau)
$$

and conclude that if $E_{\xi}(t) \equiv F_{\xi}(t, M(t))$ then $\left(E_{\xi}, \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous, $\mathbb{C}$-valued, local martingale. Thus, because $E_{\xi} \upharpoonright[0, T] \times \Omega$ is bounded for each $T>0,\left(E_{\xi}, \mathcal{F}_{t}, \mathbb{P}\right)$ is a continuous martingale. In particular,

$$
\mathbb{E}^{\mathbb{P}}\left[\exp \left(\sqrt{-1}(\xi, M(s+t)-M(s))_{\mathbb{R}^{n}}\right) \mid \mathcal{F}_{s}\right]=e^{-\frac{t}{2}|\xi|^{2}}
$$

which, together with $M(0)=0$, is enough to see that $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is a Brownian motion.

### 7.2.4. Exercises.

Exercise 7.2 .11 . Suppose that $M_{1}, M_{2} \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$, and assume that $\sigma\left(\left\{M_{1}(\tau): \tau \geq 0\right\}\right)$ is $\mathbb{P}$-independent of $\sigma\left(\left\{M_{2}(\tau): \tau \geq 0\right\}\right)$. Show that the product $M_{1} M_{2}$ is again an element of $\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$, and conclude that $\left\langle M_{1}, M_{2}\right\rangle \equiv 0 \mathbb{P}$-almost everywhere.
Exercise 7.2.12. Given $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R}), \theta \in \Theta_{\mathrm{loc}}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$, and $\eta \in$ $\Theta_{\text {loc }}^{2}\left(\left\langle I_{\theta}^{M}\right\rangle, \mathbb{P} ; \mathbb{R}\right)$, check that $\eta \theta \in \Theta_{\text {loc }}^{2}(\langle M\rangle, \mathbb{P} ; \mathbb{R})$ and that

$$
I_{\eta \theta}^{M}(t)=\int_{0}^{t} \eta(\tau) d I_{\theta}^{M}(\tau) \quad \mathbb{P} \text {-almost surely. }
$$

Exercise 7.2.13. When $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ is a Brownian motion, and therefore $\langle M\rangle(t)=t$, it is an elementary exercise to check that $\langle M\rangle(t)$ is $\mathbb{P}$-almost everywhere equal to the square variation

$$
\lim _{N \rightarrow \infty} \sum_{m=0}^{\infty}\left(M\left(t \wedge(m+1) 2^{-N}, \omega\right)-M\left(t \wedge m 2^{-N}, \omega\right)\right)^{2}
$$

of $M(\cdot, \omega) \upharpoonright[0, t]$. The purpose to this exercise is to show if $\mathbb{P}$-almost everywhere convergence is replaced by convergence in $\mathbb{P}$-probability, then the analogous result is easy to derive in general. In fact, show that, for each pair $M_{1}, M_{2} \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ and all $T \in[0, \infty)$,

$$
\begin{gathered}
\lim _{N \rightarrow \infty} \sup _{t \in[0, T]} \mid \sum_{m=0}^{\infty}\left(\left(M_{1}\left(t \wedge(m+1) 2^{-N}\right)-M_{1}\left(t \wedge m 2^{-N}\right)\right)\right. \\
\left.\times\left(M_{2}\left(t \wedge(m+1) 2^{-N}\right)-M_{2}\left(t \wedge m 2^{-N}\right)\right)\right) \\
-\left\langle M_{1}, M_{2}\right\rangle(t) \mid=0
\end{gathered}
$$

in $\mathbb{P}$-probability.
Hint: First, use polarization to reduce to the case when $M_{1}=M=M_{2}$. Next, do a little algebraic manipulation, and apply Itô 's formula to see that

$$
\begin{aligned}
\sum_{m=0}^{\infty} & \left(M\left(t \wedge(m+1) 2^{-N}, \omega\right)-M\left(t \wedge m 2^{-N}, \omega\right)\right)^{2}-\langle M\rangle(t) \\
& =2 \int_{0}^{t}\left(M(\tau)-M\left([\tau]_{N}\right)\right) d M(\tau)
\end{aligned}
$$

Finally, check that

$$
\int_{0}^{T}\left(M(\tau)-M\left([\tau]_{N}\right)\right)^{2}\langle M\rangle(d \tau) \longrightarrow 0
$$

$\mathbb{P}$-almost surely, and use this, together with Exercise 7.1.7, to get the desired conclusion.

ExERCISE 7.2.14. The following should be comforting to those who worry about such niceties. Namely, given $M \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$, set $\mathcal{A}_{t}$ equal to the $\mathbb{P}$-completion of $\sigma(\{M(\tau): \tau \in[0, t]\})$, and use the preceding exercise to see that $\langle M\rangle$ is progressively measurable with respect to $\left\{\mathcal{A}_{t}: t \geq 0\right\}$. Conclude, in particular, that no matter which filtration $\left\{\mathcal{F}_{t}: t \geq 0\right\}$ is the one with respect to which $M$ was introduced, the $\langle M\rangle$ relative $\left\{\mathcal{F}_{t}: t \geq 0\right\}$ is the same as it is relative to $\left\{\mathcal{A}_{t}: t \geq 0\right\}$.
Exercise 7.2.15. In Exercise 5.1.27, we gave a rather clumsy, and incomplete, derivation of Burkholder's Inequality. The full statement, including the extensions due to Burkholder and Gundy, is that, for each $q \in(0, \infty)$, there exist $0<c_{q}<C_{q}<\infty$ such that, for any $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$ with $M(0)=0$ and any stopping time $\zeta$,

$$
\begin{equation*}
c_{q} \mathbb{E}^{\mathbb{P}}\left[\langle M\rangle(\zeta)^{\frac{q}{2}}\right]^{\frac{1}{q}} \leq \mathbb{E}^{\mathbb{P}}\left[\|M\|_{[0, \zeta)}^{q}\right]^{\frac{1}{q}} \leq C_{q} \mathbb{E}^{\mathbb{P}}\left[\langle M\rangle(\zeta)^{\frac{q}{2}}\right]^{\frac{1}{q}} . \tag{7.2.16}
\end{equation*}
$$

Here, following A. Garsia (as recorded by Getoor and Sharpe), we will outline steps which lead to a proof (7.2.16) for $q \in[2, \infty)$. As explained in Theorem 3.1 of [16], Garsia's line of reasoning can be applied to handle the general case, but trickier arguments are required.
(i) The first step is to show that it suffices to treat the case in which both $M$ and $\langle M\rangle$ are uniformly bounded and $\zeta$ is equal to some constant $T$.
(ii) Let $q \in[2, \infty)$ be given, and set $C_{q}=\sqrt{\frac{q^{q+1}}{(q-1)^{q-1}}}$. Prove that the right hand side of (7.2.16) holds with this choice of $C_{q}$.
Hint: Begin by making the reductions in (i). Given $\epsilon>0$, set $F_{\epsilon}(x)=$ $\left(x^{2}+\epsilon^{2}\right)^{\frac{q}{2}}$, and apply Doob's Inequality plus Itô 's formula to see that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\|M\|_{[0, T]}^{q}\right] & \leq\left(q^{\prime}\right)^{q} \mathbb{E}^{\mathbb{P}}\left[F_{\epsilon}(M(T))\right] \\
& =\frac{\left(q^{\prime}\right)^{q} q(q-1)}{2} \mathbb{E}^{\mathbb{P}}\left[\int_{0}^{T} F_{\epsilon}^{\prime \prime}(M(\tau))\langle M\rangle(d \tau)\right] \\
& \leq \frac{\left(q^{\prime}\right)^{q} q(q-1)}{2} \mathbb{E}^{\mathbb{P}}\left[\left(\|M\|_{[0, T]}^{2}+\epsilon^{2}\right)^{\frac{q}{2}-1}\langle M\rangle(T)\right]
\end{aligned}
$$

Now let $\epsilon \searrow 0$, and apply Hölder's inequality.
(iii) Assume that $M$ and $\langle M\rangle$ are bounded, set $\theta(t)=\langle M\rangle(t)^{\frac{q}{4}-\frac{1}{2}}$, and take $M^{\prime}=I_{\theta}^{M}$. After noting that

$$
\left\langle M^{\prime}\right\rangle(T)=\int_{0}^{T}\langle M\rangle^{\frac{q}{2}-1}(\tau)\langle M\rangle(d \tau)=\frac{2}{q}\langle M\rangle(T)^{\frac{q}{2}},
$$

conclude that $\mathbb{E}^{\mathbb{P}}\left[\langle M\rangle(T)^{\frac{q}{2}}\right]=\frac{q}{2} \mathbb{E}^{\mathbb{P}}\left[M^{\prime}(T)^{2}\right]$.
(iv) Continuing part (iii), apply Itô's formula to see that

$$
M(T)\langle M\rangle(T)^{\frac{q}{4}-\frac{1}{2}}=M^{\prime}(T)+\int_{0}^{T} M(\tau) d\langle M\rangle(\tau)^{\frac{q}{4}-\frac{1}{2}}
$$

and conclude that $\left\|M^{\prime}\right\|_{[0, T]} \leq 2\|M\|_{[0, T]}\langle M\rangle(T)^{\frac{q}{4}-\frac{1}{2}}$. Now combine this with the result in (iii) to get the left hand side of $(7.2 .16)$ with $c_{q}=(2 q)^{-\frac{1}{2}}$. Exercise 7.2.17. Suppose that $M=\left(M_{1}, \ldots, M_{n}\right) \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})^{n}$ and set

$$
\langle\langle M\rangle\rangle(t)=\left(\left(\left\langle M_{i}, M_{j}\right\rangle(t)\right)\right)_{1 \leq i, j \leq n} .
$$

(i) Show that, $\mathbb{P}$-almost surely, $\langle\langle M\rangle\rangle(t)-\langle\langle M\rangle\rangle(s)$ is symmetric and non-negative definite for all $0 \leq s<t$. Next, set $A(t)$ equal to the trace of $\langle\langle M\rangle\rangle(t)$, and show that there exists a progressively measurable, symmetric, non-negative definite-valued function $a:[0, \infty) \times \Omega \longleftrightarrow \operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$ such that

$$
\langle\langle M\rangle\rangle(t)=\int_{0}^{t} a(\tau) A(d \tau) \quad \mathbb{P} \text {-almost surely for all } t \in[0, \infty)
$$

(ii) Referring to part (i), let $\Theta_{\text {loc }}^{2}\left(\langle\langle M\rangle\rangle, \mathbb{P} ; \mathbb{R}^{n}\right)$ be the set of progressively measurable $\theta:[0, \infty) \times \Omega \longrightarrow \mathbb{R}^{n}$ such that

$$
\int_{0}^{T}(\theta(t), a(t) \theta(t))_{\mathbb{R}^{n}} A(d t)<\infty \quad \mathbb{P} \text {-almost surely for all } T \in[0, \infty)
$$

Show that there is a unique linear map

$$
\theta \in \Theta_{\mathrm{loc}}^{2}\left(\langle\langle M\rangle\rangle, \mathbb{P} ; \mathbb{R}^{n}\right) \longmapsto I_{\theta}^{M}=\int_{0}(\theta(\tau), d M(\tau))_{\mathbb{R}^{n}} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})
$$

such that

$$
\left\langle I_{\theta}^{M}, M^{\prime}\right\rangle(d t)=\sum_{j=1}^{n} \theta_{j}(t)\left\langle M_{j}, M^{\prime}\right\rangle(d t)
$$

for all $M^{\prime} \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$.

### 7.3 Representations of Continuous Martingales

The considerations in this chapter lead to various representations of continuous local martingales in terms of Brownian motion, and this section contains some samples of these. In one way or another, all these results lend credence to the notion that there really is only one continuous local martingale: Brownian motion.
7.3.1. Representation via Random Time Change. This section is devoted to showing that if $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})$, then $M-M(0)$ is a Brownian motion run with clock $\langle M\rangle$.

In the case when, $\mathbb{P}$-almost surely, $t \rightsquigarrow\langle M\rangle(t)$ is strictly increasing and $\langle M\rangle(\infty)=\infty$, this assertion is easy to verify. Namely, one takes

$$
\begin{equation*}
\zeta(s, \omega)=\sup \{t \geq 0:\langle M\rangle(t, \omega) \leq s\} \tag{7.3.1}
\end{equation*}
$$

notes that, for each $s, \zeta(s)$ is a stopping time (in the sense described at the beginning of $\S 4.1 .2$ ), and sets $\mathcal{F}_{s}^{\prime}$ equal to the $\mathbb{P}$-completion of $\mathcal{F}_{\zeta(s)}$. Further, observe that, for $\mathbb{P}$-almost every $\omega, s \rightsquigarrow \zeta(s, \omega)$ is a strictly increasing, continuous $[0, \infty)$-valued function which satisfies

$$
\zeta(\langle M\rangle(t, \omega), \omega)=t \quad \text { and } \quad\langle M\rangle(\zeta(s, \omega), \omega))=s
$$

In particular, if

$$
\beta(s, \omega)= \begin{cases}M(\zeta(s, \omega))-M(0) & \text { when }\langle M\rangle(\infty, \omega)>s \\ 0 & \text { otherwise }\end{cases}
$$

then $\beta:[0, \infty) \times \Omega \longrightarrow \mathbb{R}$ is a $\mathbb{P}$-almost surely continuous, $\left\{\mathcal{F}_{s}^{\prime}: s \geq 0\right\}$ progressively measurable function, and $M(\cdot, \omega)-M(0)=\beta(\langle M\rangle(\cdot, \omega), \omega)$ $\mathbb{P}$-almost surely. Thus, all that remains is to check that $\left(\beta(s), \mathcal{F}_{s}^{\prime}, \mathbb{P}\right)$ is a Brownian motion. But (cf. part (iii) of Exercise 7.1.5) for each $s \in[0, \infty$ ), $\left(M(t \wedge \zeta(s))-M(0), \mathcal{F}_{t}, \mathbb{P}\right)$ is a martingale whose second moment is uniformly bounded. Hence, by Hunt's Theorem (cf. Theorem 7.1.14 in [36]), for $0 \leq$ $s_{1}<s_{2}$ and $A \in \mathcal{F}_{s_{1}}^{\prime}$,

$$
\mathbb{E}^{\mathbb{P}}\left[M\left(t \wedge \zeta\left(s_{2}\right)\right)-M\left(\zeta\left(s_{1}\right)\right), A \cap\left\{\zeta\left(s_{1}\right) \leq t\right\}\right]=0
$$

and

$$
\left.\left.\left.\left.\begin{array}{l}
\mathbb{E}^{\mathbb{P}}[M(t
\end{array}\right) \zeta\left(s_{2}\right)\right)^{2}-M\left(\zeta\left(s_{1}\right)\right)^{2}, A \cap\left\{\zeta\left(s_{1}\right) \leq t\right\}\right]\right] \text { } \quad \begin{aligned}
& \mathbb{E}^{\mathbb{P}}\left[\left(\langle M\rangle\left(t \wedge \zeta\left(s_{2}\right)\right)-s_{1}\right), A \cap\left\{\zeta\left(s_{1}\right) \leq t\right\}\right]
\end{aligned}
$$

Furthermore, $\left\|M\left(t \wedge \zeta\left(s_{2}\right)\right)-M\left(\zeta\left(s_{2}\right)\right)\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \longrightarrow 0$ and $\langle M\rangle\left(t \wedge \zeta\left(s_{2}\right)\right) \nearrow$ $s_{2} \mathbb{P}$-almost surely as $t \nearrow \infty$, which, together with the preceding, shows that $\left(\beta(s), \mathcal{F}_{s}^{\prime}, \mathbb{P}\right)$ is a continuous local martingale and that $\langle\beta\rangle(s)=s$. Now apply Lévy's Theorem (Theorem 7.2.10).

The preceding already contains the essential idea. However, there are technical difficulties which arise when $\langle M\rangle$ fails to be either strictly increasing or $\langle M\rangle(\infty)<\infty$ with positive probability. Actually, the first of these,
which brings into question the continuity of $\beta(\cdot, \omega)$, causes no real problem because, by Exercise 7.1.8, for $\mathbb{P}$-almost every $\omega, M(\cdot, \omega)$ is constant on each interval $\left[\zeta_{-}(s, \omega), \zeta(s, \omega)\right)$, where $\zeta_{-}(s, \omega) \equiv \inf \{t \geq 0:\langle M\rangle(t, \omega) \geq s\}$. A more serious issue is the one which arises when $\langle M\rangle(\infty)<\infty$. In this case the probability space may be just too anæmic to support an entire Brownian motion. For example, if $\Omega=\{\omega\}$ and $\mathcal{F}=\mathcal{F}_{t}=\{\emptyset, \Omega\}$, then there is precisely one probability measure on $(\Omega, \mathcal{F})$ and the only martingales there are constant. Thus, in general, $\langle M\rangle(\infty)<\infty$ will necessitate our supplementing the original probability space in order to obtain the desired representation in terms of a Brownian motion.
7.3.2 Theorem. Given a continuous, local martingale $\left(M(t), \mathcal{F}_{t}, \mathbb{P}\right)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a Brownian motion $\left(\beta(t), \hat{\mathcal{F}}_{t}, \hat{\mathbb{P}}\right)$ on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and a progressively measurable function $A:[0, \infty) \times \hat{\Omega} \longrightarrow[0, \infty)$ such that
(1) $A(0, \hat{\omega})=0$ and $t \rightsquigarrow A(t, \hat{\omega})$ is continuous and non-decreasing for each $\hat{\omega} \in \hat{\Omega}$,
(2) $\hat{\omega} \rightsquigarrow A(t, \hat{\omega})$ is a stopping time relative to $\left\{\hat{\mathcal{F}}_{s}: s \geq 0\right\}$ for each $t \geq 0$,
(3) the $\hat{\mathbb{P}}$-distribution of

$$
\hat{\omega} \in \hat{\Omega} \longmapsto(\beta(A(\cdot, \hat{\omega}), \hat{\omega}), A(\cdot, \hat{\omega})) \in C\left([0, \infty) ; \mathbb{R}^{2}\right)
$$

is the same as the $\mathbb{P}$-distribution of

$$
\omega \in \Omega \longmapsto(M(\cdot, \omega)-M(0, \omega),\langle M\rangle(\cdot, \omega)) \in C\left([0, \infty) ; \mathbb{R}^{2}\right)
$$

Proof: Without loss in generality, we will assume that $M(0) \equiv 0$.
Choose (cf. Exercise 7.1.6) a $\sigma(\{M(t): t \geq 0\})$-random variable $M(\infty)$ so that $M(\infty, \omega)=\lim _{t / \infty} M(t, \omega)$ for $\mathbb{P}$-almost every $\omega \in\{\langle M\rangle(\infty)<\infty\}$. Next, define $M^{\prime}:[0, \infty) \times \Omega \longrightarrow \mathbb{R}$ so that

$$
M^{\prime}(s, \omega)= \begin{cases}M(\zeta(s, \omega), \omega) & \text { if } 0 \leq s<\langle M\rangle(\infty, \omega) \\ M(\infty, \omega) & \text { if }\langle M\rangle(\infty, \omega) \leq s<\infty\end{cases}
$$

Then, by Hunt's Theorem, $\left(M^{\prime}(s), \mathcal{F}_{s}^{\prime}, \mathbb{P}\right)$ and $\left(M^{\prime}(s)^{2}-s \wedge\langle M\rangle(\infty), \mathcal{F}_{s}^{\prime}, \mathbb{P}\right)$ are martingales. In addition, by the reasoning given in the discussion above, $s \rightsquigarrow M^{\prime}(s)$ is $\mathbb{P}$-almost surely continuous. Similarly, because, for each $(t, \omega)$, the interval $[t, \zeta(\langle M\rangle(t, \omega), \omega))$ is contained in the closure of a connected component of (cf. Exercise 7.1.8) $G(\omega)$, we know that $M(\cdot, \omega)=$ $M^{\prime}(\langle M\rangle(\cdot, \omega), \omega)$ for $\mathbb{P}$-almost all $\omega$. Finally, observe that, for each $(s, t) \in$ $[0, \infty)^{2},\{\langle M\rangle(t) \leq s\}=\{\zeta(s) \geq t\} \in \mathcal{F}_{s}^{\prime}$, and conclude that $\langle M\rangle(t)$ is a
$\left\{\mathcal{F}_{s}^{\prime}: s \geq 0\right\}$-stopping time for each $t \geq 0$. Hence, when $\langle M\rangle(\infty)=\infty$ $\mathbb{P}$-almost surely, we can take $\hat{\Omega}=\Omega, \hat{\mathcal{F}}=\mathcal{F}, \hat{\mathbb{P}}=\mathbb{P}, \hat{\mathcal{F}}_{s}=\mathcal{F}_{s}^{\prime}, \beta(\cdot, \omega)=$ $M^{\prime}(\zeta(s, \omega), \omega)$, and $A(t, \omega)=\langle M\rangle(t, \omega)$.
To handle the case when $\langle M\rangle(\infty)<\infty$ with positive probability, we take $\hat{\Omega}=\Omega \times C([0, \infty) ; \mathbb{R}), \hat{\mathcal{F}}$ equal the $\mathbb{P} \times \mathbb{P}^{0}$-completion of $\mathcal{F} \times \mathcal{B}, \hat{\mathbb{P}}=\mathbb{P} \times \mathbb{P}^{0}$, and $\hat{\mathcal{F}}_{s}$ equal to the $\hat{\mathbb{P}}$-completion of $\mathcal{F}_{s}^{\prime} \times \mathcal{B}_{s}$. Then, if $B(s, \hat{\omega})=p(s)$ and $\hat{M}(s, \hat{\omega})=M^{\prime}(s, \omega)$ for $s \geq 0$ and $\hat{\omega}=(\omega, p),\left(B(s), \hat{\mathcal{F}}_{s}, \hat{\mathbb{P}}\right)$ is a Brownian motion, $\left(\hat{M}(s), \hat{\mathcal{F}}_{s}, \hat{\mathbb{P}}\right)$ is a continuous, local martingale, $\langle\hat{M}\rangle(s, \hat{\omega})=s \wedge$ $\langle M\rangle(\infty, \omega)$, and, by Exercise $7.2 .11,\langle B, \hat{M}\rangle \equiv 0$. Finally, for $\hat{\omega}=(\omega, p)$, we take $A(t, \hat{\omega})=\langle M\rangle(t, \omega)$ for $t \geq 0$ and

$$
\beta(s, \hat{\omega})= \begin{cases}\hat{M}(s, \hat{\omega}) & \text { when } 0 \leq s<\langle M\rangle(\infty, \omega) \\ B(s, \hat{\omega})-B(\langle M\rangle(\infty, \omega), \hat{\omega}) & \text { when }\langle M\rangle(\infty, \omega) \leq s<\infty\end{cases}
$$

By the reasoning in the first paragraph, we know that $A(t)$ is an $\left\{\hat{\mathcal{F}}_{s}: s \geq\right.$ $0\}$-stopping time for each $t$. Furthermore, because an equivalent description of $\beta(s, \hat{\omega})$ is to say

$$
\beta(s, \hat{\omega})=\hat{M}(s, \hat{\omega})+\int_{0}^{s} \mathbf{1}_{[A(\infty), \infty)}(\sigma) d B(\sigma)
$$

we see that $\left(\beta(s), \hat{\mathcal{F}}_{s}, \hat{\mathbb{P}}\right)$ is a continuous, local martingale with

$$
\begin{aligned}
\langle\beta\rangle(s)=\langle\hat{M}\rangle & (s)+2 \int_{0}^{s} \mathbf{1}_{[A(\infty), \infty)}(\sigma)\langle\hat{M}, B\rangle(d \sigma) \\
& +\int_{0}^{s} \mathbf{1}_{[A(\infty), \infty)}(\sigma)\langle B\rangle(d \sigma)=s \wedge A(\infty)+(s-s \wedge A(\infty))
\end{aligned}
$$

Hence, $\left(\beta(s), \hat{\mathcal{F}}_{s}, \hat{\mathbb{P}}\right)$ is a Brownian motion. Finally, by the result in the first paragraph,

$$
\beta(A(\cdot, \hat{\omega}), \hat{\omega})=M^{\prime}(\langle M\rangle(\cdot, \omega), \omega)=M(\cdot, \omega)
$$

for $\hat{\mathbb{P}}$-almost every $\hat{\omega}=(\omega, p)$, and so we have completed the proof.
One of the most important implications of Theorem 7.3.2 is the content of the following corollary.
7.3.3 Corollary. If $M \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$, then ${ }^{4}$

$$
\varlimsup_{t \rightarrow \infty} \frac{M(t)}{\sqrt{2\langle M\rangle(t) \log _{(2)}\langle M\rangle(t)}}=1=-\lim _{t \rightarrow \infty} \frac{M(t)}{\sqrt{2\langle M\rangle(t) \log _{(2)}\langle M\rangle(t)}}
$$

$\mathbb{P}$-almost surely on the set $\{\langle M\rangle(\infty)=\infty\}$. In particular, the sets $\{\langle M\rangle(\infty)$ $<\infty\}$ is $\mathbb{P}$-almost surely equal to the set of $\omega$ such that $\lim _{t \rightarrow \infty} M(t, \omega)$ exists in $\mathbb{R}$.

[^3]Proof: Using the notation in Theorem 7.3.2, what we have to do to prove the first assertion is show that

$$
\varlimsup_{t \rightarrow \infty} \frac{\beta(A(t))}{\sqrt{2 A(t) \log _{(2)} A(t)}}=1=-\lim _{t \rightarrow \infty} \frac{\beta(A(t))}{\sqrt{2 A(t) \log _{(2)} A(t)}}
$$

$\hat{\mathbb{P}}$-almost surely on the set $\{A(\infty)=\infty\}$. But, by the law of the iterated logarithm for Brownian motion (cf. Theorem 4.1.6 in [36]), this is obvious.

Given the first assertion, the second assertion follows immediately from either Exercise 7.1 .8 or by another application of the representation given by Theorem 7.3.2.
7.3.2. Representation via Stochastic Integration. Except for special cases (cf. F. Knight's Theorem in Chapter V of [27]), representation via random time change does not work when dealing with more than one $M \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ at a time. By contrast, Brownian stochastic integral representations have no dimension restriction, although they do require that the $\langle M\rangle$ 's be absolutely conditions. To make all of this precise, we will prove the following statement.
7.3.4 Theorem. Suppose that $M=\left(M_{1}, \ldots, M_{n}\right) \in\left(\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})\right)^{n}$ and that, for each $1 \leq i \leq n, t \rightsquigarrow\left\langle M_{i}\right\rangle(t)$ is $\mathbb{P}$-almost surely absolutely continuous. Then there exists progressively measurable map $\alpha:[0, \infty) \times$ $\Omega \longrightarrow \operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$ with the properties that: $\alpha(t, \omega)$ is symmetric and nonnegative definite for each $(t, \omega)$, and

$$
\left\langle M_{i}, M_{j}\right\rangle(t)=\int_{0}^{t} a_{i j}(\tau) d \tau, \quad t \geq 0, \quad \text { where } a(\tau) \equiv \alpha(\tau)^{2}
$$

Furthermore, there exist an $\mathbb{R}^{n}$-valued Brownian motion $\left(\beta(t), \hat{\mathcal{F}}_{t}, \hat{\mathbb{P}}\right)$ on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and an $\hat{\alpha} \in \Theta_{\mathrm{loc}}^{2}\left(\hat{\mathbb{P}} ; \operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)\right)$ such that the $\hat{\mathbb{P}}$-distribution of

$$
\hat{\omega} \rightsquigarrow\left(\hat{\alpha}(\cdot, \hat{\omega}), I_{\hat{\alpha}}(\cdot, \hat{\omega})\right)
$$

is the same as the $\mathbb{P}$-distribution of

$$
\omega \rightsquigarrow(\alpha(\cdot, \omega), M(\cdot, \omega)-M(0, \omega))
$$

when

$$
I_{\hat{\alpha}}(t) \equiv \int_{0}^{t} \hat{\alpha}(\tau) d \beta(\tau)
$$

Proof: The first step is to notice that, by Theorem $7.2 .1, t \rightsquigarrow\left\langle M_{i}, M_{j}\right\rangle(t)$ is $\mathbb{P}$-almost surely absolutely continuous for all $1 \leq i, j \leq n$. Hence, we can find
(cf. Theorem 5.2.26 in [36]) a progressively measurable $a:[0, \infty) \times \Omega \longrightarrow$ $\operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$ such that $\left\langle M_{i}, M_{j}\right\rangle(d t)=a_{i j}(t) d t$. Obviously, there is no reason to not take $a(t, \omega)$ to be symmetric. In addition, because

$$
\sum_{i, j=1}^{n} \xi_{i} \xi_{j}\left\langle M_{i}, M_{j}\right\rangle(d t)=\left\langle\sum_{1}^{n} \xi_{i} M_{i}\right\rangle(d t) \geq 0
$$

$a(t, \omega)$ is non-negative definite for $\lambda_{[0, \infty)} \times \mathbb{P}$-almost every $(t, \omega) \in[0, \infty)$. Hence, without loss in generality, we will take $a(t, \omega)$ to be symmetric and non-negative definite for all $(t, \omega)$.

The next step is to take $\alpha(t, \omega)$ to be the non-negative definite, symmetric square root of $a(t, \omega)$. To see that $\alpha$ is progressively measurable, we need only apply Lemma 3.2.1 to see that the non-negative, symmetric square root $\alpha^{\epsilon}(t, \omega)$ of $a(t, \omega)+\epsilon I$ is progressively measurable for all $\epsilon>0$ and then use $\alpha(t, \omega)=\lim _{\epsilon \backslash 0} \alpha^{\epsilon}(t, \omega)$.

Obviously, $\alpha(t, \omega)$ will not, in general, be invertible. Thus, we take $\pi(t, \omega)$ to denote orthogonal projection onto the null space $N(t, \omega)$ of $a(t, \omega)$ and $\alpha^{-1}(t, \omega): \mathbb{R}^{n} \longrightarrow N(t, \omega)^{\perp}$ to be the symmetric, linear map for which $N(t, \omega)$ is the null space and $\alpha^{-1}(t, \omega) \upharpoonright N(t, \omega)^{\perp}$ is the inverse of $\alpha(t, \omega) \upharpoonright$ $N(t, \omega)^{\perp}$. Again, both these maps are progressively measurable:
$\pi(t, \omega)=\lim _{\epsilon \searrow 0} a(t, \omega)(a(t, \omega)+\epsilon I)^{-1} \& \alpha^{-1}(t, \omega)=\lim _{\epsilon \searrow 0} \alpha(t, \omega)(a(t, \omega)+\epsilon I)^{-1}$.
Because (cf. Exercise 7.2.17) $\alpha^{-1}(t, \omega) \alpha(t, \omega)=\pi(t, \omega)^{\perp}$, and therefore

$$
\max _{1 \leq i \leq n} \sum_{j, j^{\prime}=1}^{n} \int_{0}^{T}\left(\alpha^{-1}\right)_{i j}(\tau)\left(\alpha^{-1}\right)_{i j^{\prime}}(\tau)\left\langle M_{j}, M_{j^{\prime}}\right\rangle(d \tau) \leq T
$$

we can take $B(t)=\int_{0}^{t} \alpha^{-1}(\tau) d M(\tau)$, in which case $\left(B(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is an $\mathbb{R}^{n}$ valued, continuous local martingale and, if $B_{\xi}(t) \equiv(\xi, B(t))_{\mathbb{R}^{n}}$, then

$$
\left\langle B_{\xi}, B_{\eta}\right\rangle(t)=\int_{0}^{t}\left(\eta, \pi(\tau)^{\perp} \xi\right)_{\mathbb{R}^{n}} d \tau
$$

Furthermore, if $X(t)=\int_{0}^{t} \alpha(\tau) d B(\tau)$, then, by Exercise 7.2.12,

$$
X_{\xi}(t) \equiv(\xi, X(t))_{\mathbb{R}^{n}}=\int_{0}^{t}(\alpha(\tau, \omega) \xi, d B(\tau))_{\mathbb{R}^{n}}=\int_{0}^{t}\left(\pi(\tau)^{\perp} \xi, d M(\tau)\right)_{\mathbb{R}^{n}}
$$

and so, if $M_{\xi}(t) \equiv(\xi, M(t))_{\mathbb{R}^{n}}$, then

$$
\left\langle X_{\xi}-M_{\xi}\right\rangle(t)=\int_{0}^{t}(\xi, \pi(\tau) a(\tau) \pi(\tau) \xi)_{\mathbb{R}^{n}} d \tau=0
$$

Clearly, in the case when $a(t, \omega)>0$ for $\lambda_{[0, \infty)} \times \mathbb{P}$-almost every $(t, \omega)$, we are done. Indeed, in this case $\left(B(t), \mathcal{F}_{t}, \mathbb{P}\right)$ is an $\mathbb{R}^{n}$-valued Brownian motion, and so we can take $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})=(\Omega, \mathcal{F}, \mathbb{P}), \hat{\mathcal{F}}_{t}=\mathcal{F}_{t}, \hat{\alpha}=\alpha$, and $\beta=B$. To handle the general case, take $\left.\hat{\Omega}=\Omega \times C\left([0, \infty) ; \mathbb{R}^{n}\right)\right), \hat{\mathcal{F}}$ and $\hat{\mathcal{F}}_{t}$ to be the $\mathbb{P} \times \mathbb{P}^{0}$-completions of $\mathcal{F} \times \mathcal{B}$ of $\mathcal{F}_{t} \times \mathcal{B}_{t}, \hat{\mathbb{P}}=\mathbb{P} \times \mathbb{P}^{0}, \hat{\alpha}(t, \hat{\omega})=\alpha(t, \omega)$, and

$$
\beta(t, \hat{\omega})=B(t, \omega)+\int_{0}^{t} \pi(\tau, \omega) d p(\tau)
$$

for $t \geq 0$ and $\hat{\omega}=(\omega, p)$. Because $\left\langle B_{\xi}\right\rangle(d t)=\pi\left|(t)^{\perp} \xi\right|^{2} d t$ and (cf. Exercise 7.2.11) $\left\langle B_{\xi}, p_{\xi}\right\rangle(d t)=0 d t$ when $p_{\xi} \equiv(\xi, p(\cdot))_{\mathbb{R}^{n}}$, it is easy to check that these choices work.

By combining the ideas in this section with those in the preceding, we arrive at the following structure theorem, which, in a somewhat different form, was anticipated by A.V. Skorohod in [31].
7.3.5 Corollary. Let $M=\left(M_{1}, \ldots, M_{n}\right) \in\left(\mathcal{M}_{\mathrm{loc}}(\mathbb{P} ; \mathbb{R})\right)^{n}$ be given, and set $A(t)=\sum_{1}^{n}\left\langle M_{i}\right\rangle(t)$. Then there is a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ on which there exists a Brownian motion $\left(\beta(t), \hat{\mathcal{F}}_{t}, \hat{\mathbb{P}}\right)$ and $\left\{\hat{\mathcal{F}}_{t}: t \geq 0\right\}$ progressively measurable maps $\hat{\alpha}:[0, \infty) \times \hat{\Omega} \longrightarrow \operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$ and $\hat{A}$ : $[0, \infty) \times \hat{\Omega} \longrightarrow[0, \infty)$ such that: $\hat{\alpha}(t, \hat{\omega})$ is symmetric, $0 I_{\mathbb{R}^{n}} \leq \hat{\alpha}(t, \hat{\omega}) \leq I_{R} n$ for all $(t, \hat{\omega}), \hat{A}(t)$ is stopping time for each $t \geq 0$, and, the $\hat{P}$-distribution of

$$
\hat{\omega} \rightsquigarrow\left(\hat{A}(t), \int_{0}^{\hat{A}(t)} \hat{\alpha}(\tau, \hat{\omega}) d \beta(\tau, \hat{\omega})\right),
$$

is the same as the $\mathbb{P}$-distribution of $\omega \rightsquigarrow(A(\cdot, \omega), M(\cdot, \omega)-M(0, \omega))$.
There are essentially no new ideas here. Namely, define $\zeta(s, \omega)=\inf \{t \geq$ $0: A(t, \omega) \geq s\}$. By the techniques used in the preceding section, we can define $M^{\prime}:[0, \infty) \times \Omega \longrightarrow \mathbb{R}^{n}$ so that $M^{\prime}(\cdot, \omega)=M(\zeta(\cdot, \omega), \omega) \mathbb{P}$-almost surely and can show that $\left(M^{\prime}(s), \mathcal{F}_{s}^{\prime}, \mathbb{P}\right)$ is an $\mathbb{R}^{n}$-valued continuous martingale when $\mathcal{F}_{s}^{\prime}$ is the $\mathbb{P}$-completion of $\mathcal{F}_{\zeta(s)}$ and that $\sum_{1}^{n}\left\langle M_{i}^{\prime}\right\rangle(d t, \omega) \leq d t$ $\mathbb{P}$-almost surely. In addition, those same techniques show that, for each $t \geq 0, A(t)$ is an $\left\{\mathcal{F}_{s}^{\prime}: s \geq 0\right\}$-stopping time and $M(t, \omega)=M^{\prime}(A(t, \omega), \omega)$ $\mathbb{P}$-almost surely. Hence, all that remains is to apply Theorem 7.3.4 to $\left(M^{\prime}(s), \mathcal{F}_{s}^{\prime}, \mathbb{P}\right)$.
A more practical reason for wanting Theorem 7.3.4 is that it enables us to prove the following sort of uniqueness theorem.
7.3.6 Corollary. Let $\sigma:[0, \infty) \times \mathbb{R}^{n} \longrightarrow \operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$ and $b:$ $[0, \infty) \times \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ be measurable functions with the properties that $\sigma(t, x)$ is
symmetric and non-negative definite for each $(t, x), t \rightsquigarrow\|\sigma(t, 0)\|_{\text {H.S. }} \vee|b(t, 0)|$ is locally bounded, and

$$
\sup _{\substack{t \in[0, T] \\ x_{2} \neq x_{1}}} \frac{\left\|\sigma\left(t, x_{2}\right)-\sigma\left(t, x_{1}\right)\right\|_{\text {H.S. }} \vee\left|b\left(t, x_{2}\right)-b\left(t, x_{1}\right)\right|}{\left|x_{2}-x_{1}\right|}<\infty
$$

for each $T>0$. Set $a(t, x)=\sigma^{2}(t, x)$, and define the time-dependent operator $t \rightsquigarrow L_{t}$ on $C^{2}\left(\mathbb{R}^{n} ; \mathbb{R}\right)$ so that

$$
L_{t} \varphi(x)=\frac{1}{2} \sum_{i, j=1}^{n} a_{i j}(t, x) \partial_{i} \partial_{j} \varphi(x)+\sum_{i=1}^{n} b_{i}(t, x) \partial_{i} \varphi(x)
$$

Then, for each $(s, x) \in[0, \infty) \times \mathbb{R}^{n}$, there is precisely one solution $\mathbb{P}_{s, x}^{L}$ to the martingale problem for $t \rightsquigarrow L_{t}$ on $C_{\mathrm{c}}^{\infty}\left(\mathbb{R}^{n} ; \mathbb{R}\right)$ starting from $x$ at time $s$. In fact, $\mathbb{P}_{s, x}^{L}$ is the $\mathbb{P}^{0}$-distribution of $p \rightsquigarrow X(\cdot,(s, x), p)$, where (cf. Theorem 5.2.2) $X(\cdot,(s, x))$ is the $\mathbb{P}^{0}$-almost surely unique, progressively measurable solution to

$$
\begin{aligned}
X(t,(s, x), p)=x & +\int_{0}^{t} \sigma(s+\tau, X(\tau,(s, x), p)) d p(\tau) \\
& +\int_{0}^{t} b(s+\tau, X(\tau,(s, x), p)) d \tau
\end{aligned}
$$

Proof: Without loss in generality, we will assume that $s=0$.
All that we have to do is show that if $\mathbb{P}$ solves the martingale problem for $t \rightsquigarrow L_{t}$ starting from $x$ at time 0 , then $\mathbb{P}$ is the $\mathbb{P}^{0}$-distribution of $p \rightsquigarrow$ $X(\cdot, x, p) \equiv X(\cdot,(s, x), p)$. Thus, suppose that $\mathbb{P}$ is a solution.

Set

$$
M(t, p) \equiv p(t)-x-\int_{0}^{t} b(\tau, p(\tau)) d \tau
$$

and observe that $\left(M(t), \overline{\mathcal{B}_{t}}, \mathbb{P}\right)$ is a continuous, local $\mathbb{R}^{n}$-valued martingale for which $\langle M\rangle(t, p)=\int_{0}^{t} a(\tau, p(\tau)) d \tau$, in the sense that $\left\langle M_{\xi}\right\rangle(t, p)=$ $\int_{0}^{t}(\xi, a(\tau, p(\tau)) \xi) d \tau$ when $M_{\xi}(t) \equiv(\xi, M(t))_{\mathbb{R}^{n}}$.

Next, determine the map $\left.\Psi:[0, \infty) \times C\left([0, \infty) ; \mathbb{R}^{n}\right)\right) \longrightarrow \mathbb{R}^{n}$ so that

$$
\Psi(t, p)=x+p(t)+\int_{0}^{t} b(\tau, \Psi(\tau, p)) d \tau, \quad t \geq 0
$$

Clearly $\Psi$ is a progressively measurable. Furthermore, $p(t)=\Psi(t, M(\cdot, p))$ and $(t, p) \rightsquigarrow \alpha(t, p) \equiv \sigma(t, \Psi(t, M(\cdot, p))$ is a progressively measurable, symmetric, non-negative definite $\operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$-valued map for which $\left\langle M_{i}, M_{j}\right\rangle(d t)$
$=\left(\alpha^{2}\right)_{i j}(t) d t$. Hence, by Theorem 7.3.4, we can find a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ on which there exists an $\mathbb{R}^{n}$-valued Brownian motion $\left(\beta(t), \hat{\mathcal{F}}_{t}, \hat{\mathbb{P}}\right)$ and a symmetric, non-negative definite valued progressively measurable function $\hat{\alpha}:[0, \infty) \times \hat{\Omega} \longrightarrow \operatorname{Hom}\left(\mathbb{R}^{n} ; \mathbb{R}^{n}\right)$ such that $\hat{\omega} \rightsquigarrow\left(\hat{\alpha}(\cdot, \hat{\omega}), I_{\hat{\alpha}}(\cdot, \hat{\omega})\right)$ has the same distribution under $\hat{\mathbb{P}}$ as $p \rightsquigarrow(\alpha(\cdot, p), M(\cdot, p))$ has under $\mathbb{P}$. In particular, if $X(t, \hat{\omega}) \equiv \Psi\left(t, I_{\hat{\alpha}}(\cdot, \hat{\omega})\right)$, then $\mathbb{P}$ is the $\hat{\mathbb{P}}$-distribution of $\hat{\omega} \rightsquigarrow X(\cdot, \hat{\omega})$ and, for each $t \geq 0, \hat{\alpha}(t)=\sigma(t, X(t)) \hat{\mathbb{P}}$-almost surely. Finally, the second of these tells us that

$$
\begin{aligned}
X(t) & =x+I_{\hat{\alpha}}(t)+\int_{0}^{t} b(\tau, X(\tau)) d \tau \\
& =x+\int_{0}^{t} \sigma(\tau, X(\tau)) d \beta(\tau)+\int_{0}^{t} b(\tau, X(\tau)) d \tau
\end{aligned}
$$

and, as we showed in Theorem 5.2.2, the solution to this equation can be written as the limit of the sequence $\left\{X_{N}\right\}_{0}^{\infty}$, where $X_{0} \equiv x$ and

$$
X_{N+1}(t)=x+\int_{0}^{t} \sigma\left(\tau, X_{N}(\tau)\right) d \beta(\tau)+\int_{0}^{t} b\left(\tau, X_{N}(\tau)\right) d \tau
$$

Since the $\hat{\mathbb{P}}$-distribution of each $X_{N}$ is uniquely determined by the distribution of $\beta$, the proof is complete.
7.3.3. Skorohod's Representation Theorem. Our final example of a representation is a particularly clever one due to A.V. Skorohod. Namely, Skorohod proved that, for any centered, square-integrable, $\mathbb{R}$-valued random variable $X$ with mean value 0 , there exists an $\mathbb{R}$-valued Brownian motion $\left(\beta(t), \mathcal{F}_{t}, \mathbb{P}\right)$ and a finite stopping time $\zeta$ such that the $\mathbb{P}$-distribution of $\omega \rightsquigarrow$ $\beta(\zeta(\omega), \omega)$ is equal to the distribution of $X$.

Skorohod's own treatment (cf. Chapter 7 in [29] and Theorem 12.4.2 in $[7])$ ) is more beautiful and direct than the one presented here. On the other hand, given the contents of the preceding subsections, our approach is more elementary. Indeed, we will use Itô's formula, in much the same way as we did in Exercise 6.3.19, to prove that there is a continuous function $u_{X}^{\prime}:[0,1) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that the $\mathbb{P}^{0}$-distribution of

$$
p \in C([0, \infty) ; \mathbb{R}) \longmapsto \int_{0}^{1} u_{X}^{\prime}(\tau, p(\tau)) d p(\tau)
$$

equals the distribution of $X$. We will then apply Theorem 7.3.2 to find a Brownian motion $\left(\beta(s), \mathcal{F}_{s}, \mathbb{P}\right)$ and a finite $\left\{\mathcal{F}_{s}: s \geq 0\right\}$-stopping time $\zeta$ such that the $\mathbb{P}$-distribution of $\omega \rightsquigarrow(\zeta(\omega), \beta(\zeta(\omega), \omega))$ is equal to the $\mathbb{P}^{0}$-distribution of

$$
p \rightsquigarrow\left(\int_{0}^{1} u_{X}^{\prime}(\tau, p(\tau))^{2} d \tau, \int_{0}^{1} u_{X}^{\prime}(\tau, p(\tau)) d p(\tau)\right) .
$$

7.3.7 Lemma. If $X$ is an $\mathbb{R}$-valued random variable with distribution function $F_{X}$ and if

$$
\left.\psi_{X}(x) \equiv \inf \left\{t \in \mathbb{R}: F_{X}(t) \geq \Gamma_{1}(-\infty, x]\right)\right\}
$$

where $\Gamma_{1}$ is the centered Gaussian measure with variance 1 , then $\psi_{X}$ is a nondecreasing, left-continuous map from $\mathbb{R}$ into itself, and the $\mathbb{P}^{0}$-distribution of $p \rightsquigarrow \psi_{X}(p(1))$ is that of $X$. Next, assume that $X$ is square-integrable. Then $\left\|\psi_{X}\right\|_{L^{2}\left(\Gamma_{1} ; \mathbb{R}\right)}=\sqrt{\mathbb{E}\left[X^{2}\right]}<\infty$. Furthermore, if $\gamma_{t}(y) \equiv(2 \pi t)^{-\frac{1}{2}} \exp \left(-\frac{y^{2}}{2 t}\right)$ is the density for the centered Gaussian measure $\Gamma_{t}$ with variance $t$, then

$$
\int_{|y-x| \geq \epsilon}\left|\psi_{X}(y)\right| \gamma_{t}(y-x) d y \leq\left\|\psi_{X}\right\|_{L^{2}\left(\Gamma_{1} ; \mathbb{R}\right)}\left(\frac{2}{t}\right)^{\frac{1}{3}} e^{x^{2}-\frac{\epsilon^{2}}{2 t}}
$$

for all $\epsilon \geq 0$ and $(t, x) \in(0,1] \times \mathbb{R}^{n}$. In particular,

$$
u_{X}(t, x) \equiv \int_{\mathbb{R}} \psi_{X}(y) \gamma_{1-t}(y-x) d y \quad \text { for }(t, x) \in[0,1) \times \mathbb{R}
$$

is well-defined. In fact, $u_{X} \in C([0,1) \times \mathbb{R} ; \mathbb{R}) \cap C^{\infty}((0,1) \times \mathbb{R} ; \mathbb{R})$, $u_{X}$ solves the backward heat equation $\left(\partial_{t}+\frac{1}{2} \partial_{x}^{2}\right) u_{X}=0$ in $(0,1) \times \mathbb{R}$, and

$$
u_{X}(t, p(t)) \longrightarrow \psi_{X}(p(1)) \quad \text { in } L^{2}\left(\mathbb{P}^{0} ; \mathbb{R}\right) \text { as } t \nearrow 1
$$

Proof: The initial assertions about $\psi_{X}$ are clear. Furthermore,

$$
\mathbb{P}^{0}\left(\psi_{X}(p(1)) \leq x\right)=\mathbb{P}^{0}\left(\Gamma_{1}((-\infty, p(1)]) \leq F_{X}(x)\right)=F_{X}(x)
$$

since $\Gamma_{1}$ is the $\mathbb{P}^{0}$-distribution of $p \rightsquigarrow p(1)$.
Next, assume that $X$ is square-integrable. Then, by the preceding, we know that $\left\|\psi_{X}\right\|_{L^{2}\left(\Gamma_{1} ; \mathbb{R}\right)}^{2}=\mathbb{E}\left[X^{2}\right]<\infty$, and, for any $t \in(0,1]$ and $k \geq 0$,

$$
\begin{aligned}
& \int_{\mathbb{R}}|y-x|^{k}\left|\psi_{X}(y)\right| \gamma_{t}(y-x) d y \leq t^{-\frac{1}{2}} e^{-\frac{x^{2}}{2}} \int_{\mathbb{R}}|y-x|^{k}\left|\psi_{X}(y)\right| e^{x y} \gamma_{1}(y) d y \\
& \quad \leq t^{-\frac{1}{2}} e^{-\frac{x^{2}}{2}}\left\|\psi_{X}\right\|_{L^{2}\left(\Gamma_{1} ; \mathbb{R}\right)}\left(\int_{\mathbb{R}}|y-x|^{2 k} e^{2 x y} \gamma_{1}(y) d y\right)^{\frac{1}{2}}
\end{aligned}
$$

When $k=0$, this proves that $u_{X}$ is well-defined as a continuous function in $[0,1) \times \mathbb{R}$. In addition, by using it for $0 \leq k \leq 2 \ell$, it is easy to justify differentiation under the integral defining $u_{X}$ and thereby prove that $u_{X}$ on $(0,1) \times \mathbb{R}$ is $2 \ell$-times differentiable with respect to $x, \ell$ times differentiable with respect to $t$, and satisfies the backward heat equation there.

Turning to the asserted estimate, first observe that

$$
\int_{|y-x| \geq \epsilon}\left|\psi_{X}(y)\right| \gamma_{t}(y-x) d y \leq\left(\int_{\mathbb{R}}\left|\psi_{X}(y)\right|^{\frac{3}{2}} \gamma_{t}(y-x) d y\right)^{\frac{2}{3}} \Gamma_{t}([-\epsilon, \epsilon] \mathbb{C})^{\frac{1}{3}}
$$

Proceeding as above, one sees that

$$
\left(\int_{\mathbb{R}}\left|\psi_{X}(y)\right|^{\frac{3}{2}} \gamma_{t}(y-x) d y\right)^{\frac{2}{3}} \leq t^{-\frac{1}{3}} e^{x^{2}}\|\psi\|_{L^{2}\left(\Gamma_{1} ; \mathbb{R}\right)}
$$

At the same time, $\Gamma_{t}([-\epsilon, \epsilon] \mathrm{C}) \leq 2 \exp \left(-\frac{\epsilon^{2}}{2 t}\right)$, which, when combined with the preceding, gives the asserted estimate. In particular, for $\epsilon>0$ and $R \in[0, \infty)$,

$$
\lim _{t \searrow 0} \sup _{|x| \leq R} \int_{|y-x| \geq \epsilon}\left|\psi_{X}(y)\right| \gamma_{t}(y-x) d y=0
$$

from which it is an easy step to see that $\lim _{t \nearrow 1} \lim _{y \rightarrow x} u_{X}(t, y)=\psi_{X}(x)$ for each $x \in \mathbb{R}$ at which $\psi_{X}$ is continuous. Thus, because $\psi_{X}$ has at most countably many points of discontinuity, this means that $\lim _{t / 1} u(t, p(t))=$ $\psi_{X}(p(1)) \mathbb{P}^{0}$-almost surely.

Since $\psi_{X}$ is locally bounded, our estimates tell us that $\sup _{t \in[0,1)}\left|u_{X}(t, \cdot)\right|$ is also locally bounded. Hence, because we already know that the convergence takes place $\mathbb{P}^{0}$-almost everywhere, we will know that $u_{X}(t, p(t)) \longrightarrow$ $\psi_{X}(p(1))$ in $L^{2}\left(\mathbb{P}^{0} ; \mathbb{R}\right)$ as $t \nearrow 1$ once we show that

$$
\lim _{R \rightarrow \infty} \sup _{t \in[0,1)} \mathbb{E}^{\mathbb{P}^{0}}\left[u_{\varphi}(t, p(t))^{2},|p(t)| \geq R\right]=0
$$

But

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{P}^{0}}\left[u_{\varphi}(t, p(t))^{2},|p(t)| \geq R\right]=\int_{|x| \geq R}\left(\int_{\mathbb{R}} \psi_{X}(y) \gamma_{1-t}(y-x) d y\right)^{2} \Gamma_{t}(d x) \\
& \quad \leq \int_{\mathbb{R}} \psi_{X}(y)^{2} f_{R}(t, y) \Gamma_{1}(d y)
\end{aligned}
$$

where

$$
\begin{aligned}
f_{R}(t, y) \equiv & \frac{e^{\frac{y^{2}}{2}}}{\sqrt{2 \pi t(1-t)}} \int_{|x| \geq R} e^{-\frac{(y-x)^{2}}{2(1-t)}} e^{-\frac{x^{2}}{2 t}} d x=\int_{S(t, R)} \gamma_{1}(\xi) d \xi \\
& \text { and } S(t, R) \equiv\left\{\xi:\left|\xi+\sqrt{\frac{t}{1-t}} y\right| \geq \frac{R}{\sqrt{t(1-t)}}\right\}
\end{aligned}
$$

Since $0 \leq f_{R}(t, y) \leq 1$ and, for each $y, \sup _{t \in[0,1)} f_{R}(t, y) \longrightarrow 0$ as $R \nearrow \infty$, we are done.

Now suppose that $X$ is a centered, square-integrable random variable. Then,

$$
u_{X}(0,0)=\int_{\mathbb{R}} \psi_{X}(y) \Gamma_{1}(d y)=\mathbb{E}^{\mathbb{P}^{0}}\left[\psi_{X}(p(1))\right]=\mathbb{E}[X]=0
$$

and so, because $u_{X}$ satisfies the backward heat equation in $(0,1) \times \mathbb{R}$, Itô's formula says that

$$
u_{X}(t, p(t))=\int_{0}^{t} u_{X}^{\prime}(\tau, p(\tau)) d p(\tau) \quad \mathbb{P}^{0} \text {-almost surely for } t \in[0,1)
$$

Hence, since $u_{X}(t, p(t)) \longrightarrow \psi_{X}(p(1))$ in $L^{2}\left(\mathbb{P}^{0} ; \mathbb{R}\right)$ as $t \nearrow 1$, we conclude that

$$
\psi_{X}(p(1))=\int_{0}^{1} u_{X}^{\prime}(\tau, p(\tau)) d p(\tau) \quad \mathbb{P}^{0} \text {-almost surely }
$$

In particular, this means that if

$$
M_{X}(t, p) \equiv \int_{0}^{t \wedge 1} u_{X}^{\prime}(\tau, p(\tau)) d p(\tau)
$$

then $\left(M_{X}(t), \overline{\mathcal{B}_{t}}, \mathbb{P}^{0}\right)$ is a square-integrable martingale for which

$$
\mathbb{E}^{\mathbb{P}^{0}}\left[\left\langle M_{X}\right\rangle(1)\right]=\mathbb{E}^{\mathbb{P}^{0}}\left[M_{X}(1)^{2}\right]=\mathbb{E}\left[X^{2}\right]
$$

In conjunction with Theorem 7.3.2, the preceding already leads to a Skorohod's representation of $X$. However, for applications, it is better to carry this line of reasoning another step before formulating it as a theorem. Namely, set

$$
\theta_{X}(t, p)= \begin{cases}u_{X}^{\prime}(t-[t], p(t)-p([t])) & \text { for } t \in[0, \infty) \backslash \mathbb{N} \\ 0 & \text { for } t \in \mathbb{N}\end{cases}
$$

Then, because $p \rightsquigarrow p(\cdot+s)-p(s)$ is $\mathbb{P}^{0}$-independent of $\overline{\mathcal{B}_{s}}$ and again has the same $\mathbb{P}^{0}$-distribution as $p$ itself, we see that the $\mathbb{P}^{0}$-distribution of

$$
\left\{\int_{0}^{m} \theta_{X}(\tau, p) d p(\tau): m \in \mathbb{Z}^{+}\right\}
$$

is the same as the distribution of the partial sums of independent copies of $X$. Thus, we have now proved the following form of Skorohod's Representation Theorem.
7.3.8 Theorem. Let $X$ be a centered, square-integrable, $\mathbb{R}$-valued random variable. Then there exists an $\mathbb{R}$-valued Brownian motion $\left(\beta(t), \mathcal{F}_{t}, \mathbb{P}\right)$ and a non-decreasing sequence $\left\{\zeta_{m}\right\}_{0}^{\infty}$ of finite stopping times such that:
(1) $\zeta_{0} \equiv 0$, the random variables $\left\{\zeta_{m}-\zeta_{m-1}: m \geq 1\right\}$ are mutually $\mathbb{P}$-independent and identically distributed, and $\mathbb{E}^{\mathbb{P}}\left[\zeta_{1}\right]=\mathbb{E}\left[X^{2}\right]$.
(2) The random variables $\left\{\beta\left(\zeta_{m}\right)-\beta\left(\zeta_{m-1}\right): m \geq 1\right\}$ are mutually $\mathbb{P}_{-}$ independent and the $\mathbb{P}$-distribution of each equals the distribution of $X$.
In fact, for each $q \in[2, \infty)$, (cf. (7.2.16))

$$
c_{q} \mathbb{E}^{\mathbb{P}}\left[\zeta_{1}^{\frac{q}{2}}\right]^{\frac{1}{q}} \leq \mathbb{E}\left[|X|^{q}\right]^{\frac{1}{q}} \leq C_{q} \mathbb{E}^{\mathbb{P}}\left[\zeta_{1}^{\frac{q}{2}}\right]^{\frac{1}{q}}
$$

Proof: There is essentially nothing left to do. Indeed, by Theorem 7.3.2, we know that there is a Brownian motion $\left(\beta(s), \mathcal{F}_{s}, \mathbb{P}\right)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a map $A:[0, \infty) \times \Omega \longrightarrow[0, \infty)$ such that $A(t)$ is an $\left\{\mathcal{F}_{s}: s \geq 0\right\}$-stopping time for each $t \geq 0$, the $\mathbb{P}$-distribution of

$$
\omega \rightsquigarrow(\beta(A(\cdot, \omega)), A(\cdot, \omega))
$$

is the same as the $\mathbb{P}^{0}$-distribution of

$$
p \rightsquigarrow\left(\int_{0}^{t} \theta_{X}(\tau, p) d p(\tau), \int_{0}^{t} \theta_{X}(\tau, p)^{2} d \tau\right) .
$$

Hence, all that remains is to set $\zeta_{m}=A(m)$.

### 7.3.4. Exercises.

Exercise 7.3.9. Let $M \in \mathcal{M}_{\text {loc }}(\mathbb{P} ; \mathbb{R})$ be given.
(i) Show that the conclusion in Exercise 7.1.6 can be strengthened to say that, for any stopping time $\zeta$,

$$
\lim _{t \rightarrow \infty} M(t \wedge \zeta) \text { exists in } \mathbb{R} \quad \mathbb{P} \text {-almost surely on }\{\langle M\rangle(\zeta)<\infty\}
$$

and
$\varlimsup_{t \rightarrow \infty} M(t \wedge \zeta)=\infty=-\lim _{t \rightarrow \infty} M(t \wedge \zeta) \quad \mathbb{P}$-almost surely on $\{\langle M\rangle(\zeta)=\infty\}$.
(ii) If $M$ is non-negative, show that $\langle M\rangle(\infty)<\infty \mathbb{P}$-almost surely.
(iii) Refer to part (i) of Exercise 6.3.12, and show that there exists a $\theta \in \Theta_{\text {loc }}^{2}\left(\mathbb{P} ; \mathbb{R}^{n}\right)$ such that

$$
M(t)=M(0)+\int_{0}^{t}(\theta(\tau), d p(\tau))_{\mathbb{R}^{n}}, \quad \mathbb{P} \text {-almost surely for each } t \in[0, \infty)
$$

Further, show that $\int_{0}^{\infty}|\theta(\tau)|^{2} d \tau<\infty \mathbb{P}$-almost surely.

ExERCISE 7.3.10. The purpose of this exercise is to see how the considerations in this section can contribute to an understanding of the relationship between dimension and explosion.
(i) Suppose that $\sigma: \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz continuous function, and let $\left(\beta(t), \mathcal{F}_{t}, \mathbb{P}\right)$ be a 1-dimensional Brownian motion. Show that, without any further conditions, the solution to the 1-dimensional stochastic differential equation

$$
d X(t)=\sigma(X(\tau)) d \beta(\tau)
$$

exists for all time whenever $\mathbb{P}(|X(0)|<\infty)=1$. That is, no matter what the distribution of $X(0)$ is or how fast $\sigma$ grows, $X(\cdot)$ will $\mathbb{P}$-almost surely not explode.
Hint: The key observation is that, because its unparameterized trajectories follow those of a 1-dimensional Brownian motion,

$$
\langle X\rangle(\infty)=\int_{0}^{\infty} \sigma^{2}(X(\tau)) d \tau=\infty \Longrightarrow X(\cdot) \text { returns to } 0 \text { infinitely often. }
$$

Thus, if $\zeta_{0} \equiv 0$ and if, for $m \geq 1$,

$$
\zeta_{m}=\left\{\begin{array}{l}
\infty \\
\inf \left\{t \geq \zeta_{m-1}: \exists \tau \in\left[\zeta_{m-1}, t\right]|X(\tau, x)|=1 \& X(t, x)=0\right\}
\end{array}\right.
$$

depending on whether $\zeta_{m-1}=\infty$ or $\zeta_{m-1}<\infty$, then, by the Markov property, $\left\{\zeta_{m+1}-\zeta_{m}: m \geq 1\right\}$ is a family of $\mathbb{P}$-mutually independent, identically distributed, strictly positive random variables. Hence, $\mathbb{P}$-almost surely, either $\langle X\rangle(\infty)<\infty$ or for all $s \geq 0$ there exists a $t \geq s$ such that $X(t)=0$. In either case, $\langle X\rangle(T)<\infty \mathbb{P}$-almost surely for all $T \geq 0$.
(ii) The analogous result holds for a diffusion in $\mathbb{R}^{2}$ associated with $L=\frac{1}{2} \alpha^{2}(x) \Delta$, where $\alpha: \mathbb{R}^{2} \longrightarrow \mathbb{R}$ is a continuously differentiable function. Namely, such a diffusion never explodes. The reason is that such a diffusion is obtained from a 2-dimensional Brownian motion by a randomtime change and that 2-dimensional Brownian motion is recurrent. Only the fact that 2-dimensional Brownian motion never actually returns to the origin complicates the argument a little. To see that recurrence is the essential property here, show that, for any $\epsilon>0$, the 3 -dimensional diffusion corresponding to $L=\frac{1}{2}\left(1+|x|^{2}\right)^{1+\epsilon} \Delta$ does explode.
Hint: Take

$$
u(x)=\frac{1}{\pi} \int_{\mathbb{R}^{3}} \frac{1}{|x-y|} \frac{1}{\left(1+|y|^{2}\right)^{1+\epsilon}} d y
$$

show that $L u=-1$, and conclude that, for $R>|x|, u(x) \geq \mathbb{E}^{\mathbb{P}_{x}^{L}}\left[\zeta_{R}\right]$, where $\zeta_{R}$ is the first exit time of $B_{\mathbb{R}^{3}}(0, R)$.

ExERCISE 7.3.11. V. Strassen made one of the most remarkable applications of Skorohod's Representation Theorem when he used it in [33] to prove a function space version of the Law of the Iterated Logarithm. Here we will aim for much less. Namely, assume the Law of the Iterated Logarithm for Brownian motion, as stated in the proof of Corollary 7.3.3, ${ }^{5}$ and prove that for any sequence $\left\{X_{m}\right\}_{0}^{\infty}$ of mutually independent, identically distributed $\mathbb{R}$-valued random variables with variance 1 ,

$$
\varlimsup_{n \rightarrow \infty} \frac{S_{m}}{\sqrt{2 m \log _{(2)} m}}=1=-\varliminf_{n \rightarrow \infty} \frac{S_{m}}{\sqrt{2 m \log _{(2)} m}}
$$

where $S_{m} \equiv \sum_{1}^{m} X_{k}$.
ExERCISE 7.3.12. Another direction in which Skorohod's Representation Theorem can be useful is in applications to Central Limit Theory. Indeed, it can be seen as providing an ingenious coupling procedure for such results. The purpose of this exercise is to give examples of this sort of application. Throughout, $\left\{X_{m}\right\}_{1}^{\infty}$ will denote a sequence of mutually independent, centered $\mathbb{R}$-valued random variables with variance $1, S_{0}=0$, and $S_{m}=\sum_{\ell=1}^{m} X_{\ell}$ for $m \geq 1$. In addition, $\left(\beta(t), \mathcal{F}_{t}, \mathbb{P}\right)$ will be the Brownian motion and $\left\{\zeta_{m}: m \geq 0\right\}$ will be the stopping times described in Theorem 7.3.8.
(i) Set $\beta^{m}(t)=m^{-\frac{1}{2}} \beta(m t)$, and note that $\left(\beta^{m}(t), \mathcal{F}_{m t}, \mathbb{P}\right)$ is again a Brownian motion and that the $\mathbb{P}$-distribution of $\beta^{m}\left(\frac{\zeta_{m}}{m}\right)$ is the same as the distribution of $m^{-\frac{1}{2}} S_{m}$. As an application of the weak laws of large numbers and the continuity of Brownian paths, conclude that

$$
\lim _{m \rightarrow \infty} \mathbb{P}\left(\left|\beta^{m}\left(\frac{\zeta_{m}}{m}\right)-\beta^{m}(1)\right| \geq \epsilon\right)=0 \quad \text { for all } \epsilon>0
$$

and use this to derive the Central Limit Theorem, which is that statement of the distribution of $m^{-\frac{1}{2}} S_{m}$ tends to the standard normal distribution.
(ii) The preceding line of reasoning can improved to give Donsker's Invariance Principal. That is, define $t \rightsquigarrow S^{m}(t)$ so that $S^{m}(0)=0, S^{m}\left(\frac{\ell}{m}\right)=$ $m^{-\frac{1}{2}} S_{m}$ and $S^{m} \upharpoonright\left[\frac{\ell-1}{m}, \frac{\ell}{m}\right]$ is linear for each $\ell \in \mathbb{Z}^{+}$. Donsker's Invariance Principal is the statement that the distribution on $C\left([0, \infty) ; \mathbb{R}^{n}\right)$ of $S^{m}(\cdot)$ tends to Wiener measure. That is, $\mathbb{E}\left[\Phi \circ S^{m}(\cdot)\right] \longrightarrow \mathbb{E}^{\mathbb{P}^{0}}[\Phi]$ for all bounded, continuous $\Phi: C([0, \infty) ; \mathbb{R}) \longrightarrow \mathbb{R}$.

[^4]To prove this, define $Y^{m}(t)$ so that $Y^{m}\left(\frac{\ell}{m}\right)=\beta^{m}\left(\frac{\zeta_{\ell}}{m}\right)$ and $Y^{m} \upharpoonright\left[\frac{\ell}{m}, \frac{\ell+1}{m}\right]$ is linear for $\ell \in \mathbb{Z}^{+}$. Note that the distribution of $Y^{m}(\cdot)$ is the same as the distribution of $S^{m}(\cdot)$, and show that, for any $L \in \mathbb{Z}^{+}, \delta>0$, and $m \geq \frac{1}{\delta}$,

$$
\left\|Y^{m}-\beta^{m}\right\|_{[0, L]} \leq 2 \sup _{\substack{0 \leq s<t \leq L \\ t-s \leq \delta}}\left|\beta^{m}(t)-\beta^{m}(s)\right|
$$

on the set where $\max _{1 \leq \ell \leq m L}\left|\frac{\zeta_{\ell}}{m}-\frac{\ell}{m}\right| \leq \delta$. Conclude that Donsker's result will follow once one shows that

$$
\lim _{m \rightarrow \infty} \mathbb{P}\left(\max _{1 \leq \ell \leq m L}\left|\frac{\zeta_{\ell}}{m}-\frac{\ell}{m}\right| \geq \delta\right)=0
$$

for each $\delta>0$.
(iii) To complete the program begun in (ii), note that it suffices to show that if $\left\{Z_{m}: m \geq 1\right\}$ is a sequence of independent, identically distributed, centered, integrable random variables, then

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{P}\left(\sup _{0 \leq \ell \leq m L}\left|\frac{1}{m} \sum_{k=1}^{\ell} Z_{k}\right| \geq \epsilon\right)=0 \tag{}
\end{equation*}
$$

for each $\epsilon>0$. When $Z_{1}$ is square integrable, (*) follows from Kolmogorov's Inequality (cf. (1.4.4) in [36]):

$$
\mathbb{P}\left(\sup _{0 \leq \ell \leq m L}\left|\frac{1}{m} \sum_{k=1}^{\ell} Z_{k}\right| \geq \epsilon\right) \leq \frac{L \operatorname{var}\left(Z_{1}\right)}{m \epsilon^{2}}
$$

To get the general case, first apply Doob's Inequality (cf. Theorem 5.2.4 in [36]) to the martingale formed by the partial sums of the $Z_{k}$ 's to see that

$$
\mathbb{P}\left(\sup _{0 \leq \ell \leq m L}\left|\frac{1}{m} \sum_{k=1}^{\ell} Z_{k}\right| \geq \epsilon\right) \leq \frac{L \mathbb{E}\left[\left|Z_{1}\right|\right]}{\epsilon}
$$

and then use this together with an approximation procedure to reduce to the case when the $Z_{k}$ 's are square integrable.


[^0]:    ${ }^{1}$ It should be recognized that A.V. Skorohod demonstrated in [30] and [31] that he already understood most of the ideas discussed below. What makes his treatment less palatable than Kunita and Watanabe's is his ignorance of the Doob-Meyer Theorem.

[^1]:    ${ }^{2}$ Remember that we have adopted $\{\zeta<t\} \in \mathcal{F}_{t}$ as the condition which determines whether $\zeta$ a stopping time.

[^2]:    ${ }^{3}$ It must be admitted that the notation here is a little confusing. Namely, we now have two closely related notations for one object: $\langle M\rangle=\langle M, M\rangle$.

[^3]:    ${ }^{4}$ Here we use $\log _{(2)} \tau$ to denote $\log (\log \tau)$ for $\tau>e$.

[^4]:    ${ }^{5}$ It should be recognized that the Law of the Iterated Logarithm for Brownian motion is essentially the same as the Law of the Iterated Logarithm for centered Gaussian random variables with variance 1 and, as such, is much easier than the statement for general centered random variables with variance 1 .

