CHEBYCHEV POLYNOMIALS

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§0: An Extremal Problem

Consider the collection $\mathfrak{P}_n$ of all nth order polynomials $P : \mathbb{R} \longrightarrow \mathbb{R}$. The starting point for what follows is the problem of finding the least upper bound on the numbers $|P'(0)|$ as P runs through $P \in \mathfrak{P}_n$ which satisfy $|P(\xi)| \le 1$ for $\xi \in [0,1]$.

After a little thought, it becomes clear that what one should do is look for the $P \in \mathfrak{P}_n$ whose values on the interval $[0,1]$ oscillate as much as possible subject to the constraint that they remain between -1 and 1. More precisely, because a non-trivial nth order polynomial can have at most n roots, what we should be looking for is a $B_n \in \mathfrak{P}_n$ which oscillates between 1 and -1 n times during the interval $[0,1]$. In the next section, we will show that such polynomials exists.

§1: The Chebychev Polynomials

Perhaps the most appealing way to introduce the family $\{C_n(\xi) : n \ge 0\}$ of *Chebychev polynomials* is to consider the problem of writing $\cos nt$ as a polynomial in $\cos t$. That is, for each $n \ge 0$, we seek a polynomial C_n such that

$$\cos nt = C_n\big(\cos t\big) \quad \text{for all } t \in \mathbb{R}. \tag{1.1}$$

Clearly, $C_0(\xi) \equiv 1$ and $C_1(\xi) = \xi$. In addition, because

$$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1,$$

we see that $C_2(\xi) = 2\xi^2 - 1$. To go further, notice that, for general $n \ge 1$,

$$\cos(n+1)t + \cos(n-1)t = \cos nt \cos t - \sin nt \sin t + \cos nt \cos t + \sin nt \sin t = 2\cos t \cos nt,$$

and so we should expect that, for any $\xi \in \mathbb{R}$,

$$C_{n+1}(\xi) = -C_{n-1}(\xi) + 2\xi C_n(\xi) \quad \text{for all } n \ge 1. \tag{1.2}$$

Lemma 1.3. *For each $\xi \in \mathbb{R}$ there is a unique sequence $\{C_n(\xi) : n \ge 1\} \subseteq \mathbb{R}$ such that $C_0(\xi) = 1$, $C_1(\xi) = 1$, and (1.2) holds for all $n \ge 1$. Moreover, for each $n \ge 0$, $C_n(\xi)$ is an nth order polynomial in ξ, $C_n(-\xi) = (-1)^n C_n(\xi)$, and (1.1) holds.*

Proof. Working by induction on $n \ge 1$, one sees that if $C_m(\xi)$ are known for $0 \le m \le n$, then $C_{n+1}(\xi)$ is uniquely determined by (1.2). Hence, by induction, the existence and uniqueness of $\{C_n(\xi) : n \geq 0\}$ present no problems. At the same time, (1.2) and induction show that $C_n(\xi)$ is an nth order polynomial and that $C_n(-\xi) = (-1)^n C_n(\xi)$. Finally, because, for any t, $\cos 0t = 1$, $\cos 1t = \cos t$, and $\cos(n+1)t = -\cos(n-1)t + 2\cos nt$, the uniqueness assertion about the $C_n(\xi)$'s with $\xi = \cos t$ leads to (1.1). $\square$

Lemma 1.3 gives us the tool with which to construct the polynomials described at the end of §0. Namely, because $C_{2n}(-\xi) = C_{2n}(\xi)$, we know that $C_{2n}(\xi) = \frac{1}{2}\big(C_{2n}(\xi) + C_{2n}(-\xi)\big)$. Hence, all the terms corresponding to odd powers of ξ are missing when $C_{2n}(\xi)$ is expressed as a polynomial in ξ, and so we can find a $B_n \in \mathfrak{P}_n$ so that $C_{2n}(\xi) = (-1)^n B_n(\xi^2)$. Moreover, by (1.1),

$$B_n\big(\sin^2 t\big) = B_n\Big(\cos^2\big(t - \tfrac{\pi}{2}\big)\Big) = (-1)^n \cos\big(2nt - n\pi\big) = \cos 2nt.$$

Hence,

$$\begin{aligned} &|B_n(\xi)| \leq 1 \quad \text{for all } \xi \in [0,1], \quad B_n\Big(\sin^2 \tfrac{m\pi}{2n}\Big) = (-1)^m \quad \text{for } 0 \leq m \leq n, \\ &\qquad \text{and } B_n\Big(\sin^2 \tfrac{(2m-1)\pi}{4n}\Big) = 0 \quad \text{for } 1 \leq m \leq n. \end{aligned} \tag{1.4}$$

§2: Back to the Extremal Problem

Having shown that they exist, we now want to show that the B_n's solve the extremal problem stated in §0; and a key role will be played by the following observation about polynomials.

Lemma 2.1. *If all the roots of $P \in \mathfrak{P}_n$ are non-negative and if $P(0) > 0$, then $P(\xi) = \sum_{m=0}^n \xi^m p_m$ where $(-1)^m p_m > 0$ for each $0 \leq m \leq n$.*

Proof. Remember that, by the fundamental theorem of algebra,

$$P(\xi) = \alpha \prod_{j=1}^n (\rho_j - \xi)$$

where $(\rho_1, \dots, \rho_n)$ are all (including multiple and complex) the roots of P.

In particular, if all the ρ_j are non-negative and $P(0) > 0$, then it is clear that α and all the ρ_j's must be positive. Hence, when we write $P(\xi) = \sum_{m=0}^n \xi^m p_m$, we see that $p_m = (-1)^m \alpha \sum_{|J|=n-m} \rho_J$ where the sum is over $n-m$ element subsets $J \subseteq \{1, \dots, n\}$ and $\rho_J = \rho_{j_1} \cdots \rho_{j_{n-m}}$ when $J = \{j_1, \dots, j_{n-m}\}$. $\square$

We[1] can now deliver the *coup de grâce*.

Theorem 2.2. *Suppose that $P \in \mathfrak{P}_n$ satisfies $|P(\xi)| \leq 1$ for all $\xi \in [0,1]$. Then, for each $0 \leq m \leq n$,*

$$\big|P^{(m)}(0)\big| \leq (-1)^m B_n^{(m)}(0), \tag{2.3}$$

[1] In truth, it was Markov who invented the beautiful line a reasoning on which we are relying here.

where $P^{(m)}$ is used to denote the mth derivative of P. In particular, $|P'(0)| \le -B_n'(0) = 2n^2$.

Proof. Let $\lambda \in (-1,1)$ be given, and consider $R = B_n - \lambda P$. Clearly R is an element of $\mathfrak{P}_n$. In addition, if $\xi_m \equiv \sin^2 \frac{m\pi}{2n}$, then $R(\xi_m) > 0$ if $0 \le m \le n$ is even and $R(\xi_m) < 0$ if $0 \le m \le n$ is odd. Hence R must have all its roots in the interval $[0,1]$, which, by Lemma 2.1, means that

$$B_n(\xi) - \lambda P(\xi) = \sum_{m=0}^{n} t^m r_m \quad \text{where } (-1)^m r_m > 0.$$

But $m! r_m = B_n^{(m)}(0) - \lambda P^{(m)}(0)$, and so we now know that $(-1)^m B_n^{(m)}(0) > (-1)^m \lambda P^{(m)}(0)$ for all $0 \le m \le n$ and $\lambda \in (-1,1)$. After letting $\lambda \nearrow 1$ if $(-1)^m P^{(m)}(0) \ge 0$ or $\lambda \searrow -1$ otherwise, we arrive at $(-1)^m B_n^{(m)}(0) \ge |P^{(m)}(0))|$.

Finally, to evaluate $B_n'(0)$, we use the relation $\cos 2nt = B_n\big(\sin^2 t\big)$ to see that

$$-2n \sin 2nt = 2 \sin t \cos t B_n'\big(\sin^2 t\big) = \sin 2t B_n\big(\sin^2 t\big),$$

which means that

$$B_n'(0) = -\lim_{t \searrow 0} \frac{2n \sin 2nt}{\sin 2t} = -2n^2. \quad \square$$

Notice that the only property of B_n which we used in the first part of the preceding proof is the existence of $0 = \xi_0 < \cdots < \xi_n \le 1$ such that $B_n(\xi_m) = (-1)^m$. That is, if Q is any element of $\mathfrak{P}_n$ for which such ξ_m exist, then $(-1)^m Q^{(m)}(0) \ge |P^{(m)}(0)|$ for all $0 \le m \le n$ and $P \in \mathfrak{P}_n$ with $|P| \le 1$ on $[0,1]$. In particular, this shows that B_n is the one and only $Q \in \mathfrak{P}_n$ with the properties that $|Q| \le 1$ on $[0,1]$ and $Q(\xi_m) = (-1)^m$ for some $0 = \xi_0 < \cdots < \xi_n \le 1$. Actually, as the following shows, we can do slightly better.

Theorem 2.4. *B_n is the one only $Q \in \mathfrak{P}_n$ with the properties that $|Q| \le 1$ on $[0,1]$ and $Q(\xi_m) = (-1)^m$ for some $0 \le \xi_0 < \cdots < \xi_n \le 1$.*

Proof. In view of the preceding remark, all that we have to do is check that $Q(0) = 1$ whenever Q has the stated properties. Thus, suppose not. That is, suppose that $Q(0) < 1$. Clearly this would mean that $\xi_0 > 0$ and that $\lambda \equiv \frac{1}{2}\big(1 + Q(0)\big) \in (-1,1)$. But this would imply that $R \equiv Q - \lambda B_n$ is an nth order polynomial which vanishes somewhere on each of the $n+1$ intervals $(0, \xi_0)$ and (ξ_{m-1}, ξ_m), $1 \le m \le n$, which is possible only if $R \equiv 0$. On the other hand, $1 = Q(\xi_0) > \lambda B_n(\xi_0)$, and so it cannot be true that $Q(0) < 1$. $\square$

Exercises:

1) Show that

$$2B_n(\xi)^2 = B_n\big(4\xi(1-\xi)\big) + 1,$$

first for all $0 \le \xi \le 1$ and then for all real ξ.

2) Compute the second derivative $B_n''(0)$ of B_n at 0, and conclude that $|P''(0)| \le \frac{4}{3} n^2(n^2 - 1)$ for any $\mathbb{P} \in \mathcal{P}_n$ satisfying $|P(\xi)| \le 1$ when $\xi \in [0,1]$.