Chapter IV
Lévy Processes

Although analysis was the engine which drove the proofs in Chapter III, probability theory can do a lot to explain the meaning of the conclusions drawn there. Specifically, in this chapter I will develop an intuitively appealing way of thinking about a random variable $\mathbf{X}$ whose distribution is infinitely divisible, an $\mathbf{X}$ for which $\mathbb{E}^{\mathbb{P}}\big[e^{\sqrt{-1}\,(\boldsymbol{\xi},\mathbf{X})_{\mathbb{R}^N}}\big]$ equals

$$\exp\Bigg(\sqrt{-1}\big(\boldsymbol{\xi},\mathbf{m}\big) - \tfrac{1}{2}\big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N}\Big[e^{\sqrt{-1}\,(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\,\mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\big(\boldsymbol{\xi},y\big)_{\mathbb{R}^N}\Big]\,M(d\mathbf{y})\Bigg)$$

for some $\mathbf{m}\in\mathbb{R}^N$, some symmetric, non-negative definite $\mathbf{C}\in\operatorname{Hom}(\mathbb{R}^N;\mathbb{R}^N)$, and Lévy measure $M\in\mathfrak{M}_2(\mathbb{R}^N)$. In most of this chapter I will deal with the case when there is no Gaussian component. That is, I will be assuming that $\mathbf{C}=0$. Because it is distinctly different, I will treat the Gaussian component separately in the final section. However, I begin with some general comments which apply to the considerations in the whole chapter.

The key idea, which seems to have been Lévy's, is to develop a dynamical picture of $\mathbf{X}$. To understand the origin of his idea, denote by $\mu\in\mathcal{I}(\mathbb{R}^N)$ the distribution of $\mathbf{X}$, and define ℓ_μ accordingly, as in Theorem 3.2.7. Then, for each $t\in[0,\infty)$, there is a unique $\mu_t\in\mathcal{I}(\mathbb{R}^N)$ for which $\widehat{\mu_t}=e^{t\ell_\mu}$, and so $\mu_{s+t}=\mu_s\star\mu_t$ for all $s,t\in[0,\infty)$. Lévy's idea was to associate with $\{\mu_t : t\ge 0\}$ a family of random variables $\{\mathbf{Z}(t) : t\ge 0\}$ which would reflect the structure of $\{\mu_t : t\ge 0\}$. Thus, for each $t\in[0,\infty)$, $\mathbf{Z}(t)$ should have distribution μ_t and, for $s,t\in[0,\infty)$, $\mathbf{Z}(s+t)-\mathbf{Z}(s)$ should be independent of $\{\mathbf{Z}(\tau) : \tau\in[0,s]\}$ and have distribution μ_t. In other words, $\{\mathbf{Z}(t) : t\ge 0\}$ should be the continuous parameter analog of the sums of independent random, identically distributed random variables. Indeed, given any $\tau>0$, let $\{\mathbf{X}_m : m\ge 0\}$ be a sequence of independent random variables with distribution μ_τ. Then $\{\mathbf{Z}(n\tau) : n\ge 0\}$ should have the same distribution as $\{\mathbf{S}_n : n\ge 0\}$, where $\mathbf{S}_n=\sum_{1\le m\le n}\mathbf{X}_m$. This observation suggests that one should think about $t\rightsquigarrow\mathbf{Z}(t)$ as a evolution which, when one understands its dynamics, will reveal information about $\mathbf{Z}(1)$ and therefore μ.

For reasons which should be now obvious, an evolution $\{\mathbf{Z}(t) : t \in [0,\infty)\}$ of the sort described above used to be called a **process with independent, homogeneous increments**, the term "process" being the common one for continuous families of random variables and the adjective "homogeneous" referring to the fact that the distribution of the increment $\mathbf{Z}(t) - \mathbf{Z}(s)$ for $0 \le s < t$ depends only on the length $t - s$ of the time interval over which it is taken. In more recent times, a process with independent, homogeneous increments is said to a **Lévy process**, and so I will adopt this more modern terminology.

Assuming that the family $\{\mathbf{Z}(t) : t \in [0,\infty)\}$ exists, notice that we already know what the joint distribution of $\{\mathbf{Z}(t_k) : k \in \mathbb{N}\}$ must be for any choice of $0 = t_0 < \cdots < t_k < \cdots$. Indeed, $\mathbf{Z}(0) = \mathbf{0}$ and

$$\mathbb{P}\big(\mathbf{Z}(t_k) - \mathbf{Z}(t_{k-1}) \in \Gamma_k,\, 1 \le k \le K\big) = \prod_{k+1}^{K} \mu_{t_k - t_{k-1}}(\Gamma_k)$$

for any $K \in \mathbb{Z}^+$ and $\Gamma_1, \dots, \Gamma_K \in \mathcal{B}_{\mathbb{R}^N}$. Equivalently, $\mathbb{P}\big(\mathbf{Z}(t_k) \in \Gamma_k,\, 0 \le k \le K\big)$ equals

$$\mathbf{1}_{\Gamma_0}(\mathbf{0}) \int \cdots \int_{(\mathbb{R}^N)^K} \prod_{k=1}^{K} \mathbf{1}_{\Gamma_k}\Bigg(\sum_{j=1}^{k} \mathbf{y}_j\Bigg) \prod_{k=1}^{K} \mu_{t_k - t_{k-1}}(d\mathbf{y}_k)$$

for any $K \in \mathbb{Z}^+$ and $\Gamma_0, \dots, \Gamma_K \in \mathcal{B}_{\mathbb{R}^N}$. My goal is this chapter is to show that each $\mu \in \mathcal{I}(\mathbb{R}^N)$ admits a Lévy process $\{\mathbf{Z}_\mu(t) : t \ge 0\}$ and that the construction of the associated Lévy process improves our understanding of μ.

Unfortunately, before I can carry out this program, I need to deal with a few technical, book keeping matters.

§4.1 Stochastic Processes, Some Generalities

Given an index $\mathcal{A}$ with some nice structure and a family $\{X(\alpha) : \alpha \in \mathcal{A}\}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in some measurable space $(E, \mathcal{B})$, it is often helpful to think about $\{X(\alpha) : \alpha \in A\}$ in terms of the map $\omega \in \Omega \longmapsto \mathbf{X}(\,\cdot\,,\omega) \in E^{\mathcal{A}}$. For instance, if $\mathcal{A}$ is linearly ordered, then $\omega \rightsquigarrow X(\,\cdot\,,\omega)$ can be thought of as random evolution. More generally, when probabilists want to indicate that they are thinking about $\{X(\alpha) : \alpha \in \mathcal{A}\}$ as the map $\omega \rightsquigarrow X(\,\cdot\,,\omega)$, they call $\{X(\alpha) : \alpha \in \mathcal{A}\}$ a **stochastic process** on $\mathcal{A}$ with **state space** $(E, \mathcal{B})$.

The **distribution of a stochastic process** is the probability measure $X_*\mathbb{P}$ on* $(E^{\mathcal{A}}, \mathcal{B}^{\mathcal{A}})$ obtained by pushing $\mathbb{P}$ forward under the map $\omega \rightsquigarrow X(\,\cdot\,,\omega)$. Hence two stochastic processes $\{X(\alpha) : \alpha \in \mathcal{A}\}$ and $\{Y(\alpha) : \alpha \in \mathcal{A}\}$ on $(E, \mathcal{B})$ have the same distribution if and only if

$$\mathbb{P}\big(X(\alpha_k) \in \Gamma_k,\, 0 \le k \le K\big) = \mathbb{P}\big(Y(\alpha_k) \in \Gamma_k,\, 0 \le k \le K\big)$$

* Recall that $\mathcal{B}^{\mathcal{A}}$ is the σ-algebra over $E^{\mathcal{A}}$ which is generated by all the maps $\psi \in E^{\mathcal{A}} \longmapsto \psi(\alpha) \in E$ as α runs over $\mathcal{A}$.

for all $K \in \mathbb{Z}^+$, $\{\alpha_0, \dots, \alpha_K\} \subseteq \mathcal{A}$, and $\Gamma_0, \dots, \Gamma_K \in \mathcal{B}$.

As long as $\mathcal{A}$ is countable, there are no problems because $E^{\mathcal{A}}$ is a reasonably tame object and $\mathcal{B}^{\mathcal{A}}$ contains lots of its subsets. However, when $\mathcal{A}$ is uncountable, $E^{\mathcal{A}}$ is a ridiculously large space and $\mathcal{B}^{\mathcal{A}}$ will be too meager to contain many of the subsets in which one is interested. The point is that for B to be in $\mathcal{B}^{\mathcal{A}}$ there must (cf. Exercise 4.1.11) be a countable subset $\{\alpha_k : k \in \mathbb{N}\}$ of $\mathcal{A}$ such that one can determine whether or not $\psi \in B$ by knowing $\{\psi(\alpha_k) : k \in \mathbb{N}\}$. Thus (cf. Exercise 4.1.11), for instance, $C\big([0,\infty);\mathbb{R}\big) \notin \mathcal{B}_{\mathbb{R}}^{[0,\infty)}$.

Probabilists expended a great deal of effort to overcome the problem raised in the preceding paragraph. For instance, using a remarkable piece of measure theoretic reasoning, J.L. Doob* proved that in the important case when $\mathcal{A} = [0,\infty)$ and $E = \mathbb{R}$, one can always make a modification, what he called the "separable modification," so that sets like $C\big([0,\infty);\mathbb{R}\big)$ become measurable. However, in recent times, probabilists have tried to simplify their lives by constructing their processes in such a way that these unpleasant measurability questions never arise. That is, if they suspect that the process should have some property which is not measurable with respect to $\mathcal{B}^{\mathcal{A}}$, they avoid constructions based on general principles, like Kolmogorov's Extension Theorem (cf. part (**iii**) of Exercise 9.1.17 below), and instead adopt a construction procedure which produces the process with the desired properties already present.

The rest of this chapter contains important examples of this approach, and the rest of this section contains a few technical preparations.

§4.1.1. The Space $D(\mathbb{R}^N)$. Unless its Lévy measure M is zero, a Lévy process for $\mu \in \mathcal{I}(\mathbb{R}^N)$ cannot be constructed so that it has continuous paths. In fact, if $M \neq 0$, then $t \rightsquigarrow \mathbf{Z}_\mu(t)$ will be almost never continuous. Nonetheless, $\{\mathbf{Z}_\mu(t) : t \geq 0\}$ can be constructed so that its paths are reasonably nice. Specifically, its paths can be made to be right continuous everywhere and have no oscillatory discontinuities. For this reason, I introduce the space $D(\mathbb{R}^N)$ of paths $\boldsymbol{\psi} : [0,\infty) \longrightarrow \mathbb{R}^N$ such that $\boldsymbol{\psi}(t) = \boldsymbol{\psi}(t+) \equiv \lim_{\tau \searrow t} \boldsymbol{\psi}(\tau)$ for each $t \in [0,\infty)$ and $\psi(t-) \equiv \lim_{\tau \nearrow t} \boldsymbol{\psi}(\tau)$ exists in $\mathbb{R}^N$ for each $t \in (0,\infty)$. Equivalently, $\boldsymbol{\psi}(0) = \boldsymbol{\psi}(0+)$, and, for each $t \in (0,\infty)$ and $\epsilon > 0$, there is a $\delta \in (0,t)$ such that $\sup\{|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(\tau)| : \tau \in (t, t+\delta)\} < \epsilon$ and $\sup\{|\boldsymbol{\psi}(t-) - \boldsymbol{\psi}(\tau)| : \tau \in (t-\delta, t)\} < \epsilon$.

The following lemma presents a few basic properties possessed by elements of $D(\mathbb{R}^N)$. In its statement, for $n \in \mathbb{N}$ and $\tau \in (0,\infty)$, $[\tau]_n^+ = \min\{m2^{-n} : m \in \mathbb{Z}^+ \text{ and } m \geq 2^n\tau\}$ and $[\tau]_n^- = [\tau]_n^+ - 2^{-n} = \max\{m2^{-n} : m \in \mathbb{N} \text{ and } m < 2^n\tau\}$. In addition, for $0 \leq a < b$,

$$\|\boldsymbol{\psi}\|_{[a,b]} \equiv \sup_{t \in [a,b]} |\boldsymbol{\psi}(t)| \tag{4.1.1}$$

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* See Chapter II of Doob's *Stochastic Processes*, published by J. Wiley

is the uniform norm of $\boldsymbol{\psi} \restriction [a,b]$, and

$$
\text{(4.1.2)} \quad \mathrm{var}_{[a,b]}(\boldsymbol{\psi}) = \sup\Bigg\{\sum_{k=1}^{K} |\boldsymbol{\psi}(t_k) - \boldsymbol{\psi}(t_{k-1})| : K \in \mathbb{Z}^+ \text{ and } a = t_0 < t_1 < \cdots < t_K = b\Bigg\}
$$

is the total variation of $\boldsymbol{\psi} \restriction [a,b]$.

Lemma 4.1.3. *If $\boldsymbol{\psi} \in D(\mathbb{R}^N)$, then, for each $t > 0$, $\|\boldsymbol{\psi}\|_{[0,t]} < \infty$, and for each $r > 0$, the set*

$$
J(t,r,\boldsymbol{\psi}) \equiv \{\tau \in (0,t] : |\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)| \ge r\}
$$

is finite subset of $(0,t]$. In addition, there exists an $n(t,r,\boldsymbol{\psi}) \in \mathbb{N}$ such that for every $n \ge n(t,r,\boldsymbol{\psi})$ and $m \in \mathbb{Z}^+ \cap (0, 2^n]$

$$
\big|\boldsymbol{\psi}\big(m2^{-n}t\big) - \boldsymbol{\psi}\big((m-1)2^{-n}t\big)\big| \ge r \implies m2^{-n} = \left[\tfrac{\tau}{t}\right]_n^+ \text{ for some } \tau \in J(t,r,\boldsymbol{\psi}).
$$

Finally,

$$
\|\boldsymbol{\psi}\|_{[0,t]} = \lim_{n\to\infty} \max\big\{|\boldsymbol{\psi}(m2^{-n}t)| : m \in \mathbb{N} \cap [0,2^n]\big\}
$$

and

$$
\mathrm{var}_{[0,t]}(\boldsymbol{\psi}) = \lim_{n\to\infty} \sum_{m \in \mathbb{Z}^+ \cap [0,2^n]} \big|\boldsymbol{\psi}\big(m2^{-n}t\big) - \boldsymbol{\psi}\big((m-1)2^{-n}t\big)\big|.
$$

Proof: Begin by noting that it suffices to treat the case when $t = 1$, since one can always reduce to this case by replacing $\boldsymbol{\psi}$ by $\tau \rightsquigarrow \boldsymbol{\psi}(t\tau)$.

If $\|\boldsymbol{\psi}\|_{[0,1]}$ were infinite, then we could find a sequence $\{\tau_n : n \ge 1\} \subseteq [0,1]$ such that $|\boldsymbol{\psi}(\tau_n)| \longrightarrow \infty$, and clearly, without loss in generality, we could choose this sequence so that $\tau_n \longrightarrow \tau \in [0,1]$ and either $\{\tau_n : n \ge 1\}$ is strictly decreasing or strictly increasing. But, in the first case this would contradict right continuity, and in the second it would contradict the existence of left limits. Thus, $\|\boldsymbol{\psi}\|_{[0,1]}$ must be finite.

Essentially the same reasoning shows that $J(1,r,\boldsymbol{\psi})$ is finite. If it were not, then we could find a sequence $\{\tau_n : n \ge 0\}$ of distinct points in $(0,1]$ such that $|\boldsymbol{\psi}(\tau_n) - \boldsymbol{\psi}(\tau_n-)| \ge r$, and again we could choose them so that either they were strictly increasing or strictly decreasing. If they were strictly increasing, then $\tau_n \nearrow \tau$ for some $\tau \in (0,1]$ and, for each $n \in \mathbb{Z}^+$, there would exist a $\tau_n' \in (\tau_{n-1}, \tau_n)$ such that $|\boldsymbol{\psi}(\tau_n) - \boldsymbol{\psi}(\tau_n')| \ge \frac{r}{2}$, which would contradict the existence of a left limit at τ. Similarly, right continuity would be contradicted if the τ_n's were decreasing.

Although it has the same flavor, the proof of the existence of $n(1,r,\boldsymbol{\psi})$ is a bit trickier. Let $0 < \tau_1 < \cdots \tau_K \le 1$ be the elements of $J(1,r,\boldsymbol{\psi})$. If $n(1,r,\boldsymbol{\psi})$

failed to exist, then we could choose a subsequence $\{(m_j, n_j) : j \geq 1\}$ from $\mathbb{Z}^+ \times \mathbb{N}$ so that $\{n_j : j \geq 1\}$ is strictly increasing, $t_j \equiv m_j 2^{-n_j} \in (0,1]$ satisfies $\big|\boldsymbol{\psi}(t_j) - \boldsymbol{\psi}(t_j - 2^{-n_j})\big| \geq r$ for all $j \in \mathbb{Z}^+$, but $t_j \neq [\tau_k]^+_{n_j}$ for any $j \in \mathbb{Z}^+$ and $1 \leq k \leq K$. If $t_j = t$ infinitely often for some t, then we would have the contradiction that $t \notin J(1, r, \boldsymbol{\psi})$ and yet $|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-)| \geq r$. Hence, I will assume that the t_j's are distinct. Further, without loss in generality, I assume that $\{t_j : j \geq 1\}$ is a subset of one of the intervals $(0, \tau_1)$, (τ_{k-1}, τ_k) for some $2 \leq k \leq K$, or of $(\tau_K, 1]$. Finally, I may and will assume that either $t_j \nearrow t \in (0,1]$ or that $t_j \searrow t \in [0,1)$. But, since $|\boldsymbol{\psi}(t_j) - \boldsymbol{\psi}(t_j - 2^{-n_j})| \geq r$, $t_j \nearrow t$ contradicts the existence of $\boldsymbol{\psi}(t-)$. Similarly, if $t_j \searrow t$ and $t_j - 2^{-n_j} \geq t$ for infinitely many $j's$, then we get a contradiction with right continuity at t. Thus, the only remaining case is when $t_j \searrow t$ and $t_j - 2^{-n_j} < t \leq t_j$ for all but a finite number of j's, in which case we get the contradiction that $t \notin J(1, r, \boldsymbol{\psi})$ and yet

$$|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-)| = \varliminf_{j\to\infty} \big|\boldsymbol{\psi}(t_j) - \boldsymbol{\psi}\big(t_j - 2^{-n_j}\big)\big| \geq r.$$

To prove the assertion about $\|\boldsymbol{\psi}\|_{[0,1]}$, simply observe that, by monotonicity, the limit exists and that, for any $t \in [0,1]$,

$$|\boldsymbol{\psi}(t)| = \lim_{n\to\infty} \big|\boldsymbol{\psi}\big([t]^+_n\big)\big| \leq \lim_{n\to\infty} \max_{0\leq m\leq 2^n} \big|\boldsymbol{\psi}(m2^{-n})\big| \leq \|\boldsymbol{\psi}\|_{[0,1]}.$$

The assertion about $\mathrm{var}_{[0,1]}(\boldsymbol{\psi})$ is proved in essentially the same, although now the monotonicity comes from the triangle inequality and the first equality in the preceding must be replaced by $|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-)| = \lim_{n\to\infty} |\boldsymbol{\psi}([t]^+_n) - \boldsymbol{\psi}([t]^-_n)|$. $\square$

I next give $D(\mathbb{R}^N)$ the topological structure corresponding to uniform convergence on compacts. Equivalently, the topological structure for which

$$\rho(\boldsymbol{\psi}, \boldsymbol{\psi}') \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{\|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_{[0,n]}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_{[0,n]}}$$

is a metric. Because it is not separable (cf. Exercise 4.1.10), this topological structure is less than ideal. Nonetheless, the metric ρ is complete. To see that it is, first observe that $|\boldsymbol{\psi}(\tau-)| \leq \|\boldsymbol{\psi}\|_{[0,t]}$ for all $0 < \tau \leq t$. Thus, if $\sup_{\ell > k} \rho(\boldsymbol{\psi}_\ell, \boldsymbol{\psi}_k) \longrightarrow 0$ as $k \to \infty$, then there exist paths $\boldsymbol{\psi} : [0,\infty) \longrightarrow \mathbb{R}^N$ and $\tilde{\boldsymbol{\psi}} : (0,\infty) \longrightarrow \mathbb{R}^N$ such that

$$\sup_{\tau\in[0,t]} |\boldsymbol{\psi}_k(\tau) - \boldsymbol{\psi}(\tau)| \longrightarrow 0 \quad \text{and} \quad \sup_{\tau\in(0,t]} |\boldsymbol{\psi}_k(\tau-) - \tilde{\boldsymbol{\psi}}(\tau)| \longrightarrow 0$$

for each $t > 0$. Therefore, if $t \geq \tau_n \searrow \tau$, then

$$\varlimsup_{n\to\infty} |\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau_n)| \leq 2\|\boldsymbol{\psi} - \boldsymbol{\psi}_k\|_{[0,t]} + \varlimsup_{n\to\infty} |\boldsymbol{\psi}_k(\tau) - \boldsymbol{\psi}_k(\tau_n)| \leq 2\|\boldsymbol{\psi} - \boldsymbol{\psi}_k\|_{[0,t]}$$

for all $k \in \mathbb{Z}^+$, and so $\boldsymbol{\psi}$ is right continuous. Essentially the same argument shows that $\boldsymbol{\psi}(\tau-) = \tilde{\boldsymbol{\psi}}(\tau)$ for $\tau > 0$, which means, of course, that $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ and that $\sup_{\tau\in(0,t]} |\boldsymbol{\psi}_k(\tau-) - \boldsymbol{\psi}(\tau-)| \longrightarrow 0$ for each $t > 0$.

One might think that I would take the measurable structure on $D(\mathbb{R}^N)$ to be the one given by the Borel field $\mathcal{B}_{D(\mathbb{R}^N)}$ determined by uniform convergence on compacts. However, this is not the choice I will make. Instead, the measurable structure I choose for $D(\mathbb{R}^N)$ is the one that $D(\mathbb{R}^N)$ inherits as a subset of $(\mathbb{R}^N)^{[0,\infty)}$. That is, the I take for $D(\mathbb{R}^N)$ the measurable structure given by the σ-algebra $\mathcal{F}_{D(\mathbb{R}^N)} = \sigma\big(\{\boldsymbol{\psi}(t) : t \in [0,\infty)\}\big)$, the σ-algebra generated by the maps $\boldsymbol{\psi} \in D(\mathbb{R}^N) \longmapsto \boldsymbol{\psi}(t) \in \mathbb{R}^N$ as t runs over $[0,\infty)$. The reason for my insisting on this choice is that I want two $D(\mathbb{R}^N)$-valued stochastic processes $\{\mathbf{X}(t) : t \geq 0\}$ and $\{\mathbf{Y}(t) : t \geq 0\}$ to induce the same measure on $D(\mathbb{R}^N)$ if they have the same distribution. Seeing as (cf. Exercise 4.1.11) $\mathcal{F}_{D(\mathbb{R}^N)} \subsetneqq \mathcal{B}_{D(\mathbb{R}^N)}$, this would not be true were I to choose the Borel structure.

Because $\mathcal{F}_{D(\mathbb{R}^N)} \neq \mathcal{B}_{D(\mathbb{R}^N)}$, $\mathcal{F}_{D(\mathbb{R}^N)}$-measurablility does not follow from topological properties like continuity. Nonetheless many functions related to the topology on $D(\mathbb{R}^N)$ are $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable. For example, the last part of Lemma 4.1.3 proves that both $\boldsymbol{\psi} \rightsquigarrow \|\boldsymbol{\psi}\|_{[0,t]}$, which is continuous, and $\boldsymbol{\psi} \rightsquigarrow \mathrm{var}_{[0,t]}(\boldsymbol{\psi})$, which is lower semicontinuous, are both $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable for all $t \in [0,\infty)$. In the next subsection, I will examine other important functions on $D(\mathbb{R}^N)$ and show that they too are $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable.

§**4.1.2. Jump functions.** Let $\mathfrak{M}_\infty(\mathbb{R}^N)$ be the space of non-negative, Borel measures M on $\mathbb{R}^N$ with the properties that $M(\{\mathbf{0}\}) = 0$ and $M\big(B(\mathbf{0},r)\complement\big) < \infty$ for all $r > 0$. A **jump function** is a map $t \in [0,\infty) \longmapsto j(t,\,\cdot\,) \in \mathfrak{M}_\infty(\mathbb{R}^N)$ with the property that, for each $\Delta \in \mathcal{B}_{\mathbb{R}^N}$ with $\mathbf{0} \notin \bar{\Delta}$, $j(0,\Delta) = 0$, $t \rightsquigarrow j(t,\Delta)$ is a non-decreasing, piecewise constant element of $D(\mathbb{R}^N)$ such that $j(t,\Delta) - j(t-,\Delta) \in \{0,1\}$ for each $t > 0$.

Lemma 4.1.4. *A map $t \rightsquigarrow j(t,\,\cdot\,)$ is a non-zero jump function if and only if there exists a set $\emptyset \neq J \subset (0,\infty)$ which is finite or countable and a set $\{\mathbf{y}_\tau : \tau \in J\} \subseteq \mathbb{R}^N \setminus \{\mathbf{0}\}$ such that $\{\tau \in J \cap (0,t] : |\mathbf{y}_\tau| \geq r\}$ is finite for each $(t,r) \in (0,\infty)^2$ and*

$$j(t,\,\cdot\,) = \sum_{\tau\in J} \mathbf{1}_{[\tau,\infty)}(t)\delta_{\mathbf{y}_\tau}. \tag{4.1.5}$$

In particular, if $t \rightsquigarrow j(t,\,\cdot\,)$ is a jump function and $t > 0$, then, either $j(t,\,\cdot\,) = j(t-,\,\cdot\,)$ or $j(t,\,\cdot\,) - j(t-,\,\cdot\,) = \delta_{\mathbf{y}}$ for some $\mathbf{y} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$.

Proof: It should be obvious that if J and $\{\mathbf{y}_\tau : \tau \in J\}$ satisfy the stated conditions, then the $t \rightsquigarrow j(t,\,\cdot\,)$ given by (4.1.5) is a jump function. To go the other direction, suppose that $t \rightsquigarrow j(t,\,\cdot\,)$ is a jump function, and, for each $r > 0$, set $f_r(t) = j\big(t, \mathbb{R}^N \setminus B(\mathbf{0},r)\big)$. Because $t \rightsquigarrow f_r(t)$ is a non-decreasing, piecewise constant, right-continuous function satisfying $f_r(0) = 0$ and $f_r(t) - f_r(t-) \in$

$\{0,1\}$ for each $t > 0$, it has at most a countable number of discontinuities, and at most $f_r(t)$ of them can occur in the interval $(0,t]$. Furthermore, if f_r has a discontinuity at τ, then $j\big(\tau, B(\mathbf{0},r)\big) - j\big(\tau-, B(\mathbf{0},r)\big) = 0$, and so the measure $\nu_\tau = j(\tau,\,\cdot\,) - j(\tau-,\,\cdot\,)$ is a $\{0,1\}$-valued probability measure on $\mathbb{R}^N$ with assigns mass 0 to $B(\mathbf{0},r)$. Hence (cf. Exercise 4.1.15) $f_r(\tau) \neq f_r(\tau-) \implies \nu_\tau = \delta_{\mathbf{y}}$ for some $\mathbf{y}_\tau \in \mathbb{R}^N \setminus B(\mathbf{0},r)$. From these considerations, it follows easily that if $J(r) = \{\tau \in (0,\infty) : f_r(\tau) \neq f_r(\tau-)\}$ and if, for each $\tau \in J(r)$, $\mathbf{y}_\tau \in \mathbb{R}^N \setminus B(\mathbf{0},r)$ is chosen so that $j(\tau,\,\cdot\,) - j(\tau-,\,\cdot\,) = \delta_{\mathbf{y}_\tau}$, then $J(r) \cap (0,t]$ is finite for all $t > 0$ and

$$j\big(t,\,\cdot\,\big) \restriction B(\mathbf{0},r)\complement = \sum_{\tau \in J(r)} \mathbf{1}_{[\tau,\infty)}(t)\delta_{\mathbf{y}_\tau}.$$

Thus, if $J = \bigcup_{r>0} J(r)$, then J is at most countable, $\{(\tau,\mathbf{y}_\tau) : \tau \in J\}$ has the required finiteness property, and (4.1.5) holds. □

The reason for my introducing jump functions is that every element $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ determines a jump function $t \rightsquigarrow j(t,\,\cdot\,,\boldsymbol{\psi})$ by the prescription

$$\begin{gathered} j(t,\Gamma,\boldsymbol{\psi}) = \sum_{\tau \in J(t,\boldsymbol{\psi})} \mathbf{1}_\Gamma\big(\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)\big), \\ \text{where } J(t,\boldsymbol{\psi}) \equiv \{\tau \in (0,t] : \boldsymbol{\psi}(\tau) \neq \boldsymbol{\psi}(\tau-)\}, \end{gathered} \tag{4.1.6}$$

for $\Gamma \subseteq \mathbb{R}^N \setminus \{\mathbf{0}\}$. To check that $j(t,\,\cdot\,,\boldsymbol{\psi})$ is well-defined and is a jump function, take $J(\boldsymbol{\psi}) = \bigcup_{t>0} J(t,\boldsymbol{\psi})$ and $\mathbf{y}_\tau = \boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)$ when $\tau \in J(\boldsymbol{\psi})$, note that, by Lemma 4.1.3, $J(\boldsymbol{\psi})$ is at most countable and that $\{(\tau,\mathbf{y}_\tau) : \tau \in J(\boldsymbol{\psi})\}$ has the finiteness required in Lemma 4.1.4, and observe that (4.1.5) holds when $j(t,\,\cdot\,) = j(t,\,\cdot\,,\boldsymbol{\psi})$ and $J = J(\boldsymbol{\psi})$.

Because it will be important for us to know that the distribution of a $D(\mathbb{R}^N)$-valued stochastic process determines the distribution of the jump functions for its paths, we will make frequent use to the following lemma.

Lemma 4.1.7. *If $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a $\mathcal{B}_{\mathbb{R}^N}$-measurable function which vanishes in a neighborhood of $\mathbf{0}$, then φ is $j(t,\,\cdot\,,\boldsymbol{\psi})$-integrable for all $(t,\boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N)$, and*

$$(t,\boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, j(t,d\mathbf{y},\boldsymbol{\psi}) \in \mathbb{R}$$

is a $\mathcal{B}_{[0,\infty)} \times \mathcal{F}_{D(\mathbb{R}^N)}$-measurable function which, for each $\boldsymbol{\psi}$, is right-continuous and piecewise constant as a function of t. Finally, for all Borel measurable $\varphi : \mathbb{R}^N \longrightarrow [0,\infty)$, $(t,\boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} \varphi(\mathbf{y}) j(t,d\mathbf{y},\boldsymbol{\psi}) \in [0,\infty]$, is $\mathcal{B}_{[0,\infty)} \times \mathcal{F}_{D(\mathbb{R}^N)}$-measurable.

Proof: The final assertion is an immediate consequence of the earlier one plus the Monotone Convergence Theorem.

Let $r > 0$ be given. If φ is a Borel measurable function which vanishes on $B(\mathbf{0},r)$, then it is immediate from the first part of Lemma 4.1.3 that φ is

$j(t,\cdot,\boldsymbol{\psi})$-integrable for all $(t,\boldsymbol{\psi})\in[0,\infty)\times D(\mathbb{R}^N)$ and, for each $\boldsymbol{\psi}\in D(\mathbb{R}^N)$ $t\rightsquigarrow\int_{\mathbb{R}^N}\varphi(\mathbf{y})\,j(t,d\mathbf{y},\boldsymbol{\psi})$ is right-continuous and piecewise constant. Thus, it suffices to show that, for each $t\in(0,\infty)$,

$$(*)\qquad \boldsymbol{\psi}\rightsquigarrow\int_{\mathbb{R}^N}\varphi(\mathbf{y})\,j(t,d\mathbf{y},\boldsymbol{\psi})\quad\text{is } \mathcal{F}_{D(\mathbb{R}^N)}\text{-measurable.}$$

Moreover, it suffices to do this when $t=1$ and φ is continuous, since rescaling time allows one to replace t by 1 and the set of φ's for which (*) is true is closed under pointwise convergence. But, by the second part of Lemma 4.1.3, we know that

$$\sum_{m=1}^{2^n}\varphi\Big(\boldsymbol{\psi}\big(m2^{-n}\big)-\boldsymbol{\psi}\big((m-1)2^{-n}\big)\Big)=\sum_{\tau\in J(1,r,\boldsymbol{\psi})}\varphi\Big(\boldsymbol{\psi}\big([\tau]_n^+\big)-\boldsymbol{\psi}\big([\tau]_n^-\big)\Big)$$

for $n\ge n(1,r,\boldsymbol{\psi})$, and therefore

$$\int_{\mathbb{R}^N}\varphi(\mathbf{y})\,j(1,d\mathbf{y},\boldsymbol{\psi})=\lim_{n\to\infty}\sum_{m=1}^{2^n}\varphi\Big(\boldsymbol{\psi}\big(m2^{-n}\big)-\boldsymbol{\psi}\big((m-1)2^{-n}\big)\Big).\quad\square$$

Here are some properties of a path $\boldsymbol{\psi}\in D(\mathbb{R}^N)$ which are determined by its relationship to its jump function. First, it should be obvious that $\boldsymbol{\psi}\in C(\mathbb{R}^N)\equiv C\big([0,\infty);\mathbb{R}^N\big)$ if and only if $j(t,\cdot,\boldsymbol{\psi})=0$ for all $t>0$. At the opposite extreme, say that a $\boldsymbol{\psi}$ is **absolutely pure jump** if and only if (cf. (3.2.13)) $j(t,\cdot,\boldsymbol{\psi})\in\mathfrak{M}_1(\mathbb{R}^N)$ and $\boldsymbol{\psi}(t)=\int\mathbf{y}\,j(t,d\mathbf{y},\boldsymbol{\psi})$ for all $t>0$. Among the absolutely pure jump paths are those which are the piecewise constant $\boldsymbol{\psi}$'s: those absolutely pure jump $\boldsymbol{\psi}$'s for which $j(t,\cdot,\boldsymbol{\psi})\in\mathfrak{M}_0(\mathbb{R}^N)$, $t>0$. Because of Lemma 4.1.7, each of these properties is $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable. In particular, if $\{\mathbf{Z}(t):t\ge0\}$ is a $D(\mathbb{R}^N)$-valued stochastic process whose paths almost surely have any one of these properties, then the paths of every $D(\mathbb{R}^N)$-valued stochastic process with the same distribution as $\{\mathbf{Z}(t):t\ge0\}$ will almost surely possess that property.

Finally, I need to address the question of when a jump function is the jump function for some $\boldsymbol{\psi}\in D(\mathbb{R}^N)$.

Theorem 4.1.8. *Let $t\rightsquigarrow j(t,\cdot)$ be a non-zero jump function, and set $j^\Gamma(t,d\mathbf{y})=\mathbf{1}_\Gamma(\mathbf{y})j(t,d\mathbf{y})$ for $\Gamma\in\mathcal{B}_{\mathbb{R}^N}$. If $\Delta\in\mathcal{B}_{\mathbb{R}^N}$ with $\mathbf{0}\notin\bar{\Delta}$ and if $\boldsymbol{\psi}^\Delta(t)=\int_\Delta\mathbf{y}\,j(t,d\mathbf{y})$, then $\boldsymbol{\psi}^\Delta$ is a piecewise constant element of $D(\mathbb{R}^N)$, $j(t,\cdot,\boldsymbol{\psi}^\Delta)=j^\Delta(t,\cdot)$, and $j(t,\cdot,\boldsymbol{\psi}-\boldsymbol{\psi}^\Delta)=j^{\mathbb{R}^N\setminus\Delta}(t,\cdot)=j(t,\cdot)-j^\Delta(t,\cdot)$ for any $\boldsymbol{\psi}\in D(\mathbb{R}^N)$ whose jump function is $t\rightsquigarrow j(t,\cdot)$. Finally, suppose that $\{\boldsymbol{\psi}_m:m\ge0\}\subseteq D(\mathbb{R}^N)$ and a non-decreasing sequence $\{\Delta_m:m\ge0\}\subseteq\mathcal{B}_{\mathbb{R}^N}$ satisfy the conditions that $\mathbb{R}^N\setminus\{\mathbf{0}\}=\bigcup_{m=0}^\infty\Delta_m$ and, for each $m\in\mathbb{N}$, $\mathbf{0}\notin\overline{\Delta_m}$ and $j(t,\cdot,\boldsymbol{\psi}_m)=j^{\Delta_m}(t,\cdot)$, $t\ge0$. If $\boldsymbol{\psi}_m\longrightarrow\boldsymbol{\psi}$ uniformly on compacts, then $j(t,\cdot,\boldsymbol{\psi})=j(t,\cdot)$, $t\ge0$.*

PROOF: Throughout the proof I will use the notation introduced in Lemma 4.1.4.

Assuming that $\mathbf{0} \notin \bar{\Delta}$, we know that

$$j^{\Delta}(t,\,\cdot\,) = \sum_{\tau \in J} \mathbf{1}_{[\tau,\infty)}(t)\mathbf{1}_{\Delta}(\mathbf{y}_\tau)\delta_{\mathbf{y}_\tau},$$

where, for each $t > 0$, there are only finitely many non-vanishing terms. At the same time,

$$\boldsymbol{\psi}^{\Delta}(t) = \sum_{\tau \in J} \mathbf{1}_{[\tau,\infty)}(t)\mathbf{1}_{\Delta}(\mathbf{y}_\tau)\mathbf{y}_\tau \text{ and } j(t,\,\cdot\,,\boldsymbol{\psi}-\boldsymbol{\psi}^{\Delta}) = \sum_{\tau \in J} \mathbf{1}_{[\tau,\infty)}(t)\mathbf{1}_{\mathbb{R}^N\setminus\Delta}(\mathbf{y}_\tau)\delta_{\mathbf{y}_\tau}$$

if $j(t,\,\cdot\,,\boldsymbol{\psi}) = j(t,\,\cdot\,)$. Thus, all that remains is to prove the final assertion. To this end, suppose that $j(t,\,\cdot\,,\boldsymbol{\psi}) \neq j(t-,\,\cdot\,,\boldsymbol{\psi})$. Since $\|\boldsymbol{\psi} - \boldsymbol{\psi}_m\|_{[0,t]} \longrightarrow 0$, there exists an m such that $\boldsymbol{\psi}_m(t) \neq \boldsymbol{\psi}_m(t-)$ and therefore that $j(t,\,\cdot\,)-j(t-,\,\cdot\,) = \delta_{\mathbf{y}}$ for some $y \in \Delta_m$. Since this means that $\boldsymbol{\psi}_n(t) - \boldsymbol{\psi}_n(t-) = \mathbf{y}$ for all $n \geq m$, it follows that $\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-) = \mathbf{y}$ and therefore that $j(t,\,\cdot\,,\boldsymbol{\psi}) - j(t-,\,\cdot\,,\boldsymbol{\psi}) = \delta_{\mathbf{y}} = j(t,\,\cdot\,) - j(t-,\,\cdot\,)$. Conversely, suppose that $j(t,\,\cdot\,) \neq j(t-,\,\cdot\,)$ and choose m so that $j(t,\,\cdot\,) - j(t-,\,\cdot\,) = \delta_{\mathbf{y}}$ for some $\mathbf{y} \in \Delta_m$. Then $\boldsymbol{\psi}_n(t) - \boldsymbol{\psi}_n(t-) = \mathbf{y}$ for all $n \geq m$. Thus, since this means that $\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-) = \mathbf{y}$, we again have that $j(t,\,\cdot\,,\boldsymbol{\psi}) - j(t-,\,\cdot\,,\boldsymbol{\psi}) = \delta_{\mathbf{y}} = j(t,\,\cdot\,) - j(t-,\,\cdot\,)$. After combining these, we see that $j(t,\,\cdot\,,\boldsymbol{\psi}) - j(t-,\,\cdot\,,\boldsymbol{\psi}) = j(t,\,\cdot\,) - j(t-,\,\cdot\,)$ for all $t > 0$, from which it is an easy step to $j(t,\,\cdot\,) = j(t,\,\cdot\,,\boldsymbol{\psi})$ for all $t \geq 0$. □

Exercises for § 4.1

EXERCISE 4.1.9. When dealing with uncountable collections of random variables, it is important to understand what functions are measurable with respect to them. To be precise, suppose that $\{X_i : i \in \mathcal{I}\}$ is a non-empty collection functions on some space Ω with values in some measurable space $(E, \mathcal{B})$, and let $\mathcal{F} = \sigma(\{X_i : i \in \mathcal{I}\})$ be the σ-algebra over Ω which they generate. Show that $A \in \mathcal{F}$ if and only if there is a sequence $\{i_m : m \in \mathbb{Z}^+\} \subseteq \mathcal{I}$ and an $\Gamma \in \mathcal{B}^{\mathbb{Z}^+}$ such that

$$A = \{\omega \in \Omega : \big(X_{i_1}(\omega), \dots, X_{i_m}(\omega), \dots\big) \in \Gamma\}.$$

More generally, if $f : \Omega \longrightarrow \mathbb{R}$, show that f is $\mathcal{F}$-measurable if and only if there is a sequence $\{i_m : m \in \mathbb{Z}^+\} \subseteq \mathcal{I}$ and a $\mathcal{F}^{\mathbb{Z}^+}$-measurable $F : E^{\mathbb{Z}^+} \longrightarrow \mathbb{R}$ such that

$$f(\omega) = F\big(X_{i_1}(\omega), \dots, X_{i_m}(\omega), \dots\big).$$

Hint: Make use of Exercise 1.1.12.

EXERCISE 4.1.10. Let $\mathbf{e} \in \mathbb{S}^{N-1}$, set $\boldsymbol{\psi}_t(\tau) = \mathbf{1}_{[t,\infty)}(\tau)\mathbf{e}$ for $t \in [0,1]$, and show that $\|\boldsymbol{\psi}_t - \boldsymbol{\psi}_s\|_{[0,1]} = 1$ for all $s \neq t$ from $[0,1]$. Conclude from this that $D(\mathbb{R}^N)$ is not separable in the topology of uniform convergence on compacts.

Exercise 4.1.11. Using Exercise 4.1.9, show that a function $\varphi : D(\mathbb{R}^N) \longrightarrow \mathbb{R}$ is $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable if and only if there exists an $(\mathbb{R}^N)^{\mathbb{N}}$-measurable function $\Phi : (\mathbb{R}^N)^{\mathbb{N}} \longrightarrow \mathbb{R}$ and a sequence $\{t_k : k \in \mathbb{N}\} \subseteq [0,\infty)$ such that

$$\varphi(\boldsymbol{\psi}) = \Phi\big(\boldsymbol{\psi}(t_0), \dots, \boldsymbol{\psi}(t_k), \dots\big), \quad \boldsymbol{\psi} \in D(\mathbb{R}^N).$$

Next, define $\boldsymbol{\psi}_t$ as in Exercise 4.1.10, and use that exercise together with the preceding to show that the open set $\{\boldsymbol{\psi} \in D(\mathbb{R}^N) : \exists\, t \in [0,1]\ \|\boldsymbol{\psi} - \boldsymbol{\psi}_t\|_{[0,1]} < 1\}$ is not $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable. Conclude that $\mathcal{B}_{D(\mathbb{R}^N)} \supsetneqq \mathcal{F}_{D(\mathbb{R}^N)}$. Similarly, conclude that neither $D(\mathbb{R}^N)$ nor $C(\mathbb{R}^N)$ is a measurable subset of $(\mathbb{R}^N)^{[0,\infty)}$. On the other hand, as we have seen, $C(\mathbb{R}^N) \in \mathcal{F}_{D(\mathbb{R}^N)}$.

Exercise 4.1.12. Show that

$$\operatorname{var}_{[0,t]}(\boldsymbol{\psi}) \ge \int_{\mathbb{R}^N} |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi}), \quad (t, \boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N). \tag{4.1.13}$$

Hint: This is most easily seen from the representation of $j(t, \cdot\,, \boldsymbol{\psi})$ in terms of point masses at the discontinuities of $\boldsymbol{\psi}$. One can use this representation to show that, for each $r > 0$,

$$\operatorname{var}_{[0,t]}(\boldsymbol{\psi}) \ge \sum_{\tau \in J(t,r,\boldsymbol{\psi})} \big|\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)\big| = \int_{|\mathbf{y}| \ge r} |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi}), \quad (t, \boldsymbol{\psi}) \in [0,\infty).$$

Exercise 4.1.14. If $\boldsymbol{\psi}$ is an absolutely pure jump path, show that $\operatorname{var}_{[0,t]}(\boldsymbol{\psi}) = \int |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi})$ and therefore that $\boldsymbol{\psi}$ has locally bounded variation. Conversely, if $\boldsymbol{\psi} \in C(\mathbb{R}^N)$ has locally bounded variation, show that $\boldsymbol{\psi}$ is an absolutely pure jump path if and only if $\operatorname{var}_{[0,t]}(\boldsymbol{\psi}) = \int |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi})$. Finally, if $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ and $j(t, \cdot\,, \boldsymbol{\psi}) \in \mathfrak{M}_1(\mathbb{R}^N)$ for all $t \ge 0$, set $\boldsymbol{\psi}_{\mathrm{c}}(t) \equiv \boldsymbol{\psi}(t) - \int \mathbf{y}\, j(t, dy, \boldsymbol{\psi})$ and show that $\boldsymbol{\psi}_{\mathrm{c}} \in C(\mathbb{R}^N)$ and

$$\operatorname{var}_{[0,t]}(\boldsymbol{\psi}) = \operatorname{var}_{[0,t]}(\boldsymbol{\psi}_{\mathrm{c}}) + \int |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi}).$$

Exercise 4.1.15. If $\nu \in \mathbf{M}_1(\mathbb{R}^N)$, show that $\nu(\Gamma) \in \{0,1\}$ for all $\Gamma \in \mathcal{B}_{\mathbb{R}^N}$ if and only if $\nu = \delta_{\mathbf{y}}$ for some $\mathbf{y} \in \mathbb{R}^N$.

Hint: Begin by showing that it suffices to handle the case when $N = 1$. Next, assuming that $N = 1$, show that ν is compactly supported, let m be its mean value, and show that $\nu = \delta_m$.

§ 4.2 Discontinuous Lévy Processes

In this section I will construct the Lévy processes corresponding to those $\mu \in \mathcal{I}(\mathbb{R}^N)$ with no Gaussian component. That is,

$$\begin{aligned}\hat{\mu}(\boldsymbol{\xi}) = \exp\Bigg(&\sqrt{-1}\big(\boldsymbol{\xi}, \mathbf{m}_\mu\big)_{\mathbb{R}^N} \\ &+ \int_{\mathbb{R}^N} \Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})} - 1 - \sqrt{-1}\, \mathbf{1}_{[0,1]}(|\mathbf{y}|)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M_\mu(d\mathbf{y})\Bigg).\end{aligned} \tag{4.2.1}$$

Because they are the building blocks out of which all such processes are made, I will treat separately the case when μ is a Poisson measure π_M for some $M \in \mathfrak{M}_0(\mathbb{R}^N)$ and will call the corresponding Lévy process the **Poisson process** associated with M.

§ **4.2.1. The Simple Poisson Process.** I begin with the case when $N = 1$ and $M = \delta_1$, for which π_M is the **simple Poisson measure** $e^{-1}\sum_{m=0}^{\infty}\frac{1}{m!}\delta_m$ whose Fourier transform is $\exp\Big(e^{\sqrt{-1}\xi} - 1\Big)$.

To construct the Poisson process associated with δ_1, start with a sequence $\{\tau_m : m \geq 1\}$ of independent, **unit exponential** random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is,

$$\mathbb{P}\big(\{\omega : \tau_1(\omega) > t_1, \dots, \tau_n(\omega) > t_n\}\big) = \exp\left(-\sum_{m=1}^{n} t_m^+\right)$$

for all $n \in \mathbb{Z}^+$ and $(t_1, \dots, t_n) \in \mathbb{R}^n$. Without loss in generality, I may and will assume that $\tau_m(\omega) > 0$ for all $m \in \mathbb{Z}^+$ and $\omega \in \Omega$. In addition, by the Strong Law of Large Numbers, I may and will assume that $\sum_{m=1}^{\infty}\tau_m(\omega) = \infty$ for all $\omega \in \Omega$. Next, set $T_0(\omega) = 0$ and $T_n(\omega) = \sum_{m=1}^{n}\tau_m(\omega)$, and define

$$\text{(4.2.2)}\quad N(t,\omega) = \max\{n \in \mathbb{N} : T_n(\omega) \leq t\} = \sum_{n=1}^{\infty}\mathbf{1}_{[T_n(\omega),\infty)}(t) \quad \text{for } t \in [0,\infty).$$

Clearly $t \rightsquigarrow N(t,\omega)$ is a non-decreasing, right-continuous, piecewise constant path, $\mathbb{N}$-valued path which starts at 0 and, whenever it jumps, jumps by $+1$. In particular, $N(\,\cdot\,,\omega) \in D(\mathbb{R}^N)$, $N(t,\omega) - N(t-,\omega) \in \{0,1\}$ for all $t \in (0,\infty)$, and $j\big(t,\,\cdot\,,N(\,\cdot\,,\omega)\big) = N(t,\omega)\delta_1$.

Because $\mathbb{P}\big(N(t) = n\big) = \mathbb{P}\big(T_n \leq t < T_{n+1}\big)$, $\mathbb{P}\big(N(t) = 0\big) = \mathbb{P}(\tau_1 > t) = e^{-t}$, and, when $n \geq 1$, (here $|\Gamma|$ denotes the Lebesgue measure of $\Gamma \in \mathcal{B}_{\mathbb{R}^n}$)

$$\mathbb{P}\big(N(t) = n\big) = \int\!\!\cdots\!\!\int_A e^{-\sum_{m=1}^{n+1}\tau_m}\, d\tau_1 \cdots d\tau_{n+1} = e^{-t}|B|,$$

where $A = \big\{(\tau_1, \dots, \tau_{n+1}) \in (0,\infty)^{n+1} : \sum_{m=1}^{n}\tau_m \leq t < \sum_{m=1}^{n+1}\tau_m\big\}$ and $B = \big\{(\tau_1, \dots, \tau_n) \in (0,\infty)^n : \sum_{m=1}^{n}\tau_m \leq t\big\}$. By making the change of variables $s_m = \sum_{j=1}^{m}\tau_j$ and remarking that the associated Jacobian is 1, one sees that $|B| = |C|$, where $C = \big\{(s_1, \dots, s_n) \in \mathbb{R}^n : 0 < s_1 < \cdots < s_n \leq t\big\}$. Since $|C| = \frac{t^n}{n!}$, we have shown that the $\mathbb{P}$-distribution of $N(t)$ is the Poisson measure $\pi_{t\delta_1}$. In particular, π_{δ_1} is the $\mathbb{P}$-distribution of $N(1)$.

I now want to use the same sort of calculation to show that $\{N(t) : t \in [0,\infty)\}$ is a **simple Poisson process**, that is, a Lévy process for π_{δ_1}. See Exercise 4.2.18 for another, perhaps preferable, approach.

Lemma 4.2.3. *For any $(s,t) \in [0,\infty)$, the $\mathbb{P}$-distribution of the increment $N(s+t)-N(s)$ is $\pi_{t\delta_1}$. In addition, for any $K \in \mathbb{Z}^+$ and $0 = t_0 < t_1 < \cdots < t_K$, the increments $\{N(t_k) - N(t_{k-1}) : 1 \le k \le K\}$ are independent.*

Proof: What I have to show is that for all $K \in \mathbb{Z}^+$, $0 = n_0 \le \cdots \le n_K$, and $0 = t_0 < t_1 < \cdots < t_K$,

$$\mathbb{P}\big(N(t_k) - N(t_{k-1}) = n_k - n_{k-1},\, 1 \le k \le K\big) = \prod_{k=1}^{K} \frac{e^{-(t_k - t_{k-1})}(t_k - t_{k-1})^{n_k - n_{k-1}}}{(n_k - n_{k-1})!},$$

which is equivalent to checking that

$$\mathbb{P}\big(N(t_k) = n_k,\, 1 \le k \le K\big) = \prod_{k=1}^{K} \frac{e^{-(t_k - t_{k-1})}(t_k - t_{k-1})^{n_k - n_{k-1}}}{(n_k - n_{k-1})!};$$

and, since the case when $n_K = 0$ is trivial, I will assume that $n_K \ge 1$. In fact, because neither side is changed if one removes those n_k's for which $n_k = n_{k-1}$, I will assume the $0 = n_0 < \cdots < n_K$.

Begin by noting that

$$\begin{aligned}\mathbb{P}\big(N(t_k) = n_k,\, 0 \le k \le K\big) &= \mathbb{P}\big(T_{n_k} \le t_k < T_{n_{k+1}},\, 1 \le k \le K\big) \\ &= \int_A \cdots \int e^{-\sum_{m=1}^{n_K+1} \tau_m}\, d\tau_1 \cdots d\tau_{n_K+1} = e^{-t_K}|B|,\end{aligned}$$

where

$$A = \left\{(\tau_1, \dots, \tau_{n_K+1}) \in (0,\infty)^{n_K+1} : \sum_{m=1}^{n_k} \tau_m \le t_k < \sum_{m=1}^{n_k+1} \tau_m,\, 1 \le k \le K\right\}$$

and

$$B = \left\{(\tau_1, \dots, \tau_{n_K}) \in (0,\infty)^{n_K} : t_{k-1} < \sum_{m=1}^{n_k} \tau_m \le t_k : 1 \le k \le K\right\}.$$

To compute $|B|$, make the change of variables $s_m = \sum_{j=1}^{m} \tau_j$ to see that $|B| = |C|$, where

$$C = \big\{(s_1, \dots, s_{n_K}) \in \mathbb{R}^{n_K} : t_{k-1} < s_{n_{k-1}+1} < \cdots < s_{n_k} \le t_k \text{ for } 1 \le k \le K\big\}.$$

Finally, for $1 \le k \le K$, set

$$C_k = \big\{(s_{n_{k-1}+1}, \dots, s_{n_k}) \in \mathbb{R}^{n_k - n_{k-1}} : t_{k-1} < s_{n_{k-1}+1} < \cdots < s_{n_k} \le t_k\big\},$$

and check that

$$e^{-t_K}|C| = e^{-t_K}\prod_{k\in S}|C_k| = e^{-t_K}\prod_{k\in S}\frac{(t_k-t_{k-1})^{n_k-n_{k-1}}}{(n_k-n_{k-1})!}$$
$$= \prod_{k=1}^{K}\frac{e^{-(t_k-t_{k-1})}(t_k-t_{k-1})^{n_k-n_{k-1}}}{(n_k-n_{k-1})!}. \quad \square$$

The simple Poisson process $\{N(t) : t \geq 0\}$ is aptly named. It starts at 0, waits a unit exponential holding time before jumping to 1, sits at 1 for another, independent, unit exponential holding time before jumping to 2, etc. Thus, since π_{δ_1} is the distribution of this process at time 1, we now have an appealing picture of the way in which simple Poisson random variables arise.

Given $\alpha \in [0,\infty)$, I will say that a $D(\mathbb{R})$-valued process whose distribution is the same as that of $\{N(\alpha t) : t \geq 0\}$ is a **simple Poisson process run at rate** α.

§ **4.2.2. Compound Poisson Processes.** I next want to build a Poisson process associated with a general $M \in \mathfrak{M}_0(\mathbb{R}^N)$. If $M = 0$, there is nothing to do, since the corresponding process will simply sit at $\mathbf{0}$ for all time. If $M \neq 0$, I write it as $\alpha\nu$, where $\alpha = M(\mathbb{R}^N)$ and $\nu = \frac{M}{\alpha}$. After augmenting our probability space if necessary, we introduce a sequence $\{\mathbf{X}_n : n \geq 1\}$ of independent, ν-distributed, random variables which are independent of the unit exponential random variables $\{\tau_m : m \geq 1\}$ out of which we built the simple Poisson process $\{N(t) : t \geq 0\}$ in the preceding subsection. Further, since $M(\{\mathbf{0}\}) = 0$, I may and will assume that none of the $\mathbf{X}_n$'s is ever $\mathbf{0}$. Finally, set

$$\mathbf{Z}_M(t,\omega) = \sum_{1\leq n\leq N(\alpha t,\omega)} \mathbf{X}_n(\omega), \tag{4.2.4}$$

with the understanding that a sum over the empty set is $\mathbf{0}$.

Clearly, the process $\{\mathbf{Z}_M(t) : t \geq 0\}$ is nearly as easily understood as is the simple Poisson process. Like the simple Poisson process, its paths are right-continuous, start at $\mathbf{0}$, and are piecewise constant. Further, its holding times and jumps are all independent of one another. The difference is that its holding times are now α-exponential random variable (i.e., exponential with mean value $\frac{1}{\alpha}$) and its jumps are random variables with distribution ν. In particular,

$$j\big(t,\,\cdot\,,\mathbf{Z}_M(\,\cdot\,,\omega)\big) = \sum_{1\leq n\leq N(\alpha t,\omega)} \delta_{\mathbf{X}_n(\omega)} = \sum_{n=1}^{\infty} \mathbf{1}_{[T_n(\omega),\infty)}(t)\delta_{\mathbf{X}_n(\omega)}. \tag{4.2.5}$$

I now want to check that $\{\mathbf{Z}_M(t) : t \geq 0\}$ is a Lévy process for π_M, and, as such, deserves to be called a Poisson process associated with M: the one with

rate $M(\mathbb{R}^N)$ and **jump distribution** $\frac{M}{M(\mathbb{R}^N)}$. That is, I want to show that, for each $0 = t_0 < t_1 < \cdots t_K$, the random variables $\mathbf{Z}_M(t_k) - \mathbf{Z}_M(t_{k-1})$, $1 \le k \le K$, are independent and that the kth one has distribution $\pi_{(t_k - t_{k-1})M}$. Equivalently, I need to check that, for any $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_K \in \mathbb{R}^N$,

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{K}\big(\boldsymbol{\xi}_k, \mathbf{Z}_M(t_k) - \mathbf{Z}_M(t_{k-1})\big)_{\mathbb{R}^N}\right)\right] = \prod_{k=1}^{K}\widehat{\pi_{\tau_k M}}(\boldsymbol{\xi}_k),$$

where $\tau_k = t_k - t_{k-1}$. But, because of our independence assumptions, the above expectation is equal to

$$\begin{aligned}\sum_{n_K \ge \cdots \ge n_1 \ge 0} & \mathbb{P}\big(N(\alpha t_k) - N(\alpha t_{k-1}) = n_k - n_{k-1},\, 1 \le k \le K\big) \\ & \times \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{K}\sum_{n_{k-1}+1 < m \le n_k}\big(\boldsymbol{\xi}_k, \mathbf{X}_m\big)_{\mathbb{R}^N}\right)\right] \\ = \sum_{n_K \ge \cdots \ge n_1 \ge 0}\prod_{k=1}^{K} & \frac{e^{-\alpha\tau_k}\tau_k^{n_k - n_{k-1}}}{(n_k - n_{k-1})!}\hat{\nu}(\boldsymbol{\xi})^{n_k - n_k - 1} = \prod_{k=1}^{K}\widehat{\pi_{\tau_k M}}(\boldsymbol{\xi}_k).\end{aligned}$$

Any stochastic process $\{\mathbf{Z}(t) : t \ge 0\}$ with right-continuous, piecewise constant paths and the same distribution as the process $\{\mathbf{Z}_M(t) : t \ge 0\}$ just constructed is called a **Poisson process** associated with M.

Here is a beautiful and important procedure for transforming one Poisson process into another.

LEMMA 4.2.6. *Suppose that $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$ is a Borel measurable function which takes the origin in $\mathbb{R}^N$ into the origin in $\mathbb{R}^{N'}$, and, for $M \in \mathfrak{M}_0(\mathbb{R}^N)$, define $M^F \in \mathfrak{M}_0(\mathbb{R}^{N'})$ by*

$$M^F(\Gamma) = M\Big(F^{-1}\big(\Gamma \setminus \{\mathbf{0}\}\big)\Big) \quad \text{for } \Gamma \in \mathcal{B}_{\mathbb{R}^{N'}}.$$

If $\{\mathbf{Z}(t) : t \ge 0\}$ is a Poisson process associated with π_M and

$$\mathbf{Z}^F(t,\omega) = \int_{\mathbb{R}^N} F(\mathbf{y})\, j\big(t, d\mathbf{y}, \mathbf{Z}(\,\cdot\,,\omega)\big) \quad \text{for } (t,\omega) \in [0,\infty) \times \Omega, \tag{4.2.7}$$

then $\{\mathbf{Z}^F(t) : t \ge 0\}$ is a Poisson associated with π_{M^F}. Moreover, if, for each i in an index set $\mathcal{I}$, $F_i : \mathbb{R}^N \longrightarrow \mathbb{R}^{N_i}$ is a Borel measurable satisfying $F_i(\mathbf{0}) = \mathbf{0}$ and, for each $\mathbf{y} \in \mathbb{R}^N$, there is at most one $i \in \mathcal{I}$ for which $F_i(\mathbf{y}) \ne \mathbf{0}$, then the processes $\big\{\{\mathbf{Z}^{F_i}(t) : t \ge 0\} : i \in \mathcal{I}\big\}$ are independent.

Proof: In proving the first part, I will, without loss in generality, assume that (cf. (4.2.4)) $\mathbf{Z} = \mathbf{Z}_M$. But then, by (4.2.5),

$$\mathbf{Z}^F(t,\omega) = \sum_{1 \le n \le N(\alpha t,\omega)} F\big(\mathbf{X}_n(\omega)\big),$$

from which the first assertion is immediate by the same computation with which I just showed that $\{\mathbf{Z}_M(t) : t \ge 0\}$ is a Poisson process associated with M.

To prove the second assertion, I begin by observing that it suffices to treat the case when $\mathcal{I} = \{1,2\}$. To see this, suppose that we know the result in that case, and let $n > 2$ and a set $\{i_1, \dots, i_n\}$ of distinct elements from $\mathcal{I}$ be given. By taking $F_1 = (F_{i_1}, \dots, F_{i_{n-1}})$, $F_2 = F_{i_n}$, and applying the assumed result, we would have that $\{\mathbf{Z}^{F_{i_n}}(t) : t \ge 0\}$ is independent of $\big\{\big(\mathbf{Z}^{F_{i_1}}(t), \dots, \mathbf{Z}^{F_{i_{n-1}}}(t)\big) : t \ge 0\big\}$. Hence, proceeding by induction, we would be able to show that the processes $\big\{\{\mathbf{Z}^{F_{i_m}}(t) : t \ge 0\} : 1 \le m \le n\big\}$ are independent.

Now assume that $\mathcal{I} = \{1,2\}$. What we have to check is that, for any $K \in \mathbb{Z}^+$, $0 = t_0 < t_1 < \cdots < t_K$ and $\{(\boldsymbol{\xi}_k^1, \boldsymbol{\xi}_k^2) : 1 \le k \le K\} \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Bigg[\exp\Bigg(\sqrt{-1}\sum_{k=1}^{K}\Big[\big(\boldsymbol{\xi}_k^1, \mathbf{Z}^{F_1}(t_k) - \mathbf{Z}^{F_1}(t_{k-1})\big)_{\mathbb{R}^{N_1}} \\
&\qquad\qquad + \big(\boldsymbol{\xi}_k^2, \mathbf{Z}^{F_2}(t_k) - \mathbf{Z}^{F_2}(t_{k-1})\big)_{\mathbb{R}^{N_2}}\Big]\Bigg)\Bigg] \\
&\quad = \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{K}\big(\boldsymbol{\xi}_k^1, \mathbf{Z}^{F_1}(t_k) - \mathbf{Z}^{F_1}(t_{k-1})\big)_{\mathbb{R}^{N_1}}\right)\right] \\
&\qquad\qquad \times \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{K}\big(\boldsymbol{\xi}_k^2, \mathbf{Z}^{F_2}(t_k) - \mathbf{Z}^{F_2}(t_{k-1})\big)_{\mathbb{R}^{N_2}}\right)\right].
\end{aligned}$$

For this purpose, take $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N_1+N_2}$ to be given by $F(\mathbf{y}) = \big(F_1(\mathbf{y}), F_2(\mathbf{y})\big)$, and set $\boldsymbol{\xi}_k = (\boldsymbol{\xi}_k^1, \boldsymbol{\xi}_k^2)$. Then the first expression in the preceding equals

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{K}\big(\boldsymbol{\xi}_k, \mathbf{Z}^F(t_k) - \mathbf{Z}^F(t_{k-1})\big)_{\mathbb{R}^{N_1+N_2}}\right)\right] \\
&\quad = \prod_{k=1}^{K}\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\sqrt{-1}\big(\boldsymbol{\xi}_k, \mathbf{Z}^F(t_k - t_{k-1})\big)_{\mathbb{R}^{N_1+N_2}}\Big)\Big],
\end{aligned}$$

since $\{\mathbf{Z}^F(t) : t \ge 0\}$ has independent, homogeneous increments. Hence, it suffices for us to observe that, for any $t > 0$ and $\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \boldsymbol{\xi}^2)$,

$$\begin{aligned}\mathbb{E}^P\Big[\exp\Big(\big(\boldsymbol{\xi}, \mathbf{Z}^F(t)\big)_{\mathbb{R}^{N_1+N_2}}\Big)\Big] &= \exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi}, F(\mathbf{y}))_{\mathbb{R}^{N_1+N_2}}} - 1\Big)\,M(d\mathbf{y})\right)\\ &= \exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi}^1, F_1(\mathbf{y}))_{\mathbb{R}^{N_1}}} - 1\Big)\,M(d\mathbf{y})\right)\\ &\qquad\times \exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi}^2, F_2(\mathbf{y}))_{\mathbb{R}^{N_2}}} - 1\Big)\,M(d\mathbf{y})\right)\\ &= \mathbb{E}^P\Big[\exp\Big(\big(\boldsymbol{\xi}^1, \mathbf{Z}^{F_1}(t)\big)_{\mathbb{R}^{N_1}}\Big)\Big]\mathbb{E}^P\Big[\exp\Big(\big(\boldsymbol{\xi}^2, \mathbf{Z}^{F_2}(t)\big)_{\mathbb{R}^{N_2}}\Big)\Big]. \quad \square\end{aligned}$$

As an essentially immediate consequence of Lemma 4.2.6 and Theorem 4.1.8, we have the following important conclusion.

THEOREM 4.2.8. *If $\{\mathbf{Z}(t) : t \geq 0\}$ is a Poisson process associated with π_M, then, for each $\Delta \in \mathcal{B}_{\mathbb{R}^N\setminus\{\mathbf{0}\}}$, $\big\{j\big(t, \Delta, \mathbf{Z}(\,\cdot\,)\big) : t \geq 0\big\}$ is a simple Poisson process run at rate $M(\Delta)$. Moreover, if*

$$\mathbf{Z}^\Delta(t) = \int_\Delta \mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}\big) \text{ and } M^\Delta(\Gamma) = M(\Delta\cap\Gamma) \text{ for } \Gamma \in \mathcal{B}_{\mathbb{R}^N},$$

then $\{\mathbf{Z}^\Delta(t) : t \geq 0\}$ is the Poisson process associated with M^Δ and $j\big(t, \Gamma, \mathbf{Z}^\Delta\big) = j\big(t, \Gamma\cap\Delta, \mathbf{Z}\big)$ for all $(t,\Gamma) \in [0,\infty)\times\mathcal{B}_{\mathbb{R}^N}$. Finally, if $\{\Delta_i : i \in \mathcal{I}\}$ is a family of mutually disjoint Borel subsets of $\mathbb{R}^N\setminus\{\mathbf{0}\}$, then both the Poisson processes $\big\{\{\mathbf{Z}^{\Delta_i}(t) : t \geq 0\} : i \in \mathcal{I}\big\}$ as well as the jump processes $\big\{\{j\big(t, \Delta_i, \mathbf{Z}\big) : t \geq 0\} : i \in \mathcal{I}\big\}$ are mutually independent.

The result in Theorem 4.2.8 says that the jumps of Poisson process can be decomposed into a family of mutually independent, simple Poisson process run at rates determined by the M-measure of the jump sizes. The next result can be thought of as a re-assembly procedure which complements this decomposition result.

THEOREM 4.2.9. *If $\big\{\{\mathbf{Z}_k(t) : t \geq 0\} : 1 \leq k \leq K\big\}$ are independent Poisson processes associated with $\{M_k : 1 \leq k \leq K\} \subseteq \mathfrak{M}_0(\mathbb{R}^N)$, then*

$$\left\{\mathbf{Z}(t) \equiv \sum_{k=1}^K \mathbf{Z}_k(t) : t \geq 0\right\} \text{ is a Poisson process associated with } M \equiv \sum_{k=1}^K M_k.$$

Next, suppose that the M_k's are mutually singular in the sense that, for each k, there exists a $\Delta_k \in \mathcal{B}_{\mathbb{R}^N\setminus\{\mathbf{0}\}}$ such that the Δ_k's are mutually disjoint and $M_k\big(\Delta_k\complement\big) = 0 = M_\ell(\Delta_k)$ for $\ell \neq k$. Then, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$j\big(t, \cdot\,, \mathbf{Z}(\,\cdot\,,\omega)\big) = \sum_{k=1}^K j\big(t, \cdot\,, \mathbf{Z}_k(\,\cdot\,,\omega)\big), \quad t \in [0,\infty).$$

Equivalently, for $\mathbb{P}$-almost every $\omega \in \Omega$ and all $t \geq 0$, there is at most one k such that $\mathbf{Z}_k(t,\omega) \neq \mathbf{Z}_k(t-,\omega)$.

PROOF: Clearly, $\{\mathbf{Z}(t) : t \geq 0\}$ starts at $\mathbf{0}$ and has independent increments. In addition, for any $s,\, t \in [0,\infty)$ and $\boldsymbol{\xi} \in \mathbb{R}^N$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{Z}(s+t)-\mathbf{Z}(s))_{\mathbb{R}^N}}\Big] &= \prod_{k=1}^{K}\mathbb{E}^{\mathbb{P}}\Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{Z}_k(s+t)-\mathbf{Z}_k(s))_{\mathbb{R}^N}}\Big] \\ &= \prod_{k=1}^{K}\exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}}-1\Big)\,M_k(d\mathbf{y})\right) \\ &= \exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}}-1\Big)\,M(d\mathbf{y})\right).\end{aligned}$$

Now assume that the M_k's are as in the final part of the statement, and choose Δ_k's accordingly. Without loss in generality, I will assume that $\mathbb{R}^N \setminus \{\mathbf{0}\} = \bigcup_{k=1}^{K}\Delta_k$. Also, because the assertion depends only on the joint distribution of the processes involved, I may and will assume that

$$\mathbf{Z}_k(t) = \int_{\Delta_k}\mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}\big) \quad \text{for } 1 \leq k \leq K,$$

since then $\mathbf{Z}(t) = \sum_{k=1}^{K}\mathbf{Z}_k(t)$, and, by Theorem 4.2.8, the $\mathbf{Z}_k$'s are independent and the kth one is a Poisson process associated with M_k. But with this choice, another application of Theorem 4.2.8 shows that $j\big(t,\Gamma,\mathbf{Z}_k\big) = j\big(t,\Gamma\cap\Delta_k,\mathbf{Z}\big)$, and therefore

$$j\big(t,\Gamma,\mathbf{Z}\big) = \sum_{k=1}^{K} j\big(t,\Gamma,\mathbf{Z}_k\big), \quad t \in [0,\infty). \quad \square$$

Because the paths of a Poisson process are piecewise constant, they certainly have finite variation on each compact time interval. The first part of next lemma allows us to estimate that variation. The estimate in the second part will be used in § 4.2.5.

LEMMA 4.2.10. *If* $\{\mathbf{Z}(t) : t \geq 0\}$ *is a Poisson process associated with* $M \in \mathfrak{M}_0(\mathbb{R}^N)$, *then*

$$\mathbb{E}^{\mathbb{P}}\big[\mathrm{var}_{[0,t]}(\mathbf{Z})\big] = t\int_{\mathbb{R}^N}|\mathbf{y}|\, M(dy).$$

In addition, if $\int_{\mathbb{R}^N}|\mathbf{y}|\,M(d\mathbf{y}) < \infty$ *and* $\bar{\mathbf{Z}}(t) = \mathbf{Z}(t) - \int_{\mathbb{R}^N}\mathbf{y}\,M(d\mathbf{y})$, *then*

$$\mathbb{P}\big(\|\bar{\mathbf{Z}}\|_{[0,t]} \geq R\big) \leq \frac{Nt}{R^2}\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{Z}}(t)|^2\big] = \frac{Nt}{R^2}\int_{\mathbb{R}^N}|\mathbf{y}|^2\, M(d\mathbf{y}).$$

PROOF: Again I will assume that (cf. (4.2.4)) $\mathbf{Z} = \mathbf{Z}_M$, in which case

$$\mathrm{var}_{[0,t]}(\mathbf{Z}) = \sum_{1\leq m\leq N(\alpha t)}|\mathbf{X}_m|.$$

Hence (cf. the notation used in § 4.1.1)

$$\mathbb{E}^{\mathbb{P}}\big[\mathrm{var}_{[0,t]}(\mathbf{Z})\big] = \mathbb{E}^{\mathbb{P}}\big[N(\alpha t)\big]\mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_1|\big] = \alpha t\int_{\mathbb{R}^N}|\mathbf{y}|\,\nu(d\mathbf{y}) = t\int_{\mathbb{R}^N}|\mathbf{y}|\,M(d\mathbf{y}).$$

Turning to the second part, begin by observing that

$$\begin{aligned}\mathbb{P}\big(\|\bar{\mathbf{Z}}\|_{[0,t]} > R\big) &= \lim_{n\to\infty}\mathbb{P}\left(\max_{1\le m\le 2^n}\big|\bar{\mathbf{Z}}\big(m2^{-n}t\big)\big| > R\right)\\ &\le N\lim_{n\to\infty}\sup_{\mathbf{e}\in\mathbb{S}^{N-1}}\mathbb{P}\left(\max_{1\le m\le 2^n}\big|\big(\mathbf{e},\bar{\mathbf{Z}}\big(m2^{-n}t\big)\big)_{\mathbb{R}^N}\big| > R\right).\end{aligned}$$

Next, given $\mathbf{e}\in\mathbb{S}^{N-1}$ and $n\ge 1$, write

$$\big(\mathbf{e},\bar{\mathbf{Z}}(m2^{-n}t)\big)_{\mathbb{R}^N} = \sum_{1\le\ell\le m}\big(\mathbf{e},\bar{\mathbf{Z}}(\ell 2^{-n}t) - \bar{\mathbf{Z}}((\ell-1)2^{-n}t)\big)_{\mathbb{R}^N},$$

and apply Kolmogorov's Inequality to conclude that

$$\mathbb{P}\left(\max_{1\le m\le 2^n}\big|\big(\mathbf{e},\bar{\mathbf{Z}}\big(m2^{-n}t\big)\big)_{\mathbb{R}^N}\big| > R\right) \le R^{-2}\mathbb{E}^{\mathbb{P}}\big[\big(\mathbf{e},\bar{\mathbf{Z}}(t)\big)^2_{\mathbb{R}^N}\big].$$

Thus, we will be done once we check that $\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{Z}}_M(t)|^2\big] = t\int_{\mathbb{R}^N}|\mathbf{y}|^2\,M(d\mathbf{y})$. To this end, first note that $\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{Z}}(t)|^2\big] = \mathbb{E}^{\mathbb{P}}\big[|\mathbf{Z}(t)|^2\big] - \alpha^2t^2|\mathbf{m}|^2$, where $\mathbf{m} = \int_{\mathbb{R}^N}\mathbf{y}\,\nu(d\mathbf{y})$. At the same time, if $\bar{\mathbf{X}}_m = \mathbf{X}_m - \mathbf{m}$, then $\mathbb{E}^{\mathbb{P}}\big[|\mathbf{Z}(t)|^2\big]$ equals

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left|\sum_{1\le m\le N(\alpha t)}\mathbf{X}_m\right|^2\right] &= \mathbb{E}^{\mathbb{P}}\left[\left|\sum_{1\le m\le N(\alpha t)}\bar{\mathbf{X}}_m\right|^2\right] + |\mathbf{m}|^2\mathbb{E}^{\mathbb{P}}\big[N(\alpha t)^2\big]\\ &= \alpha t\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{X}}_1|^2\big] + |\mathbf{m}|^2\big(\alpha^2t^2+\alpha t\big) = \alpha t\mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_1|^2\big] + \alpha^2t^2|\mathbf{m}|^2.\end{aligned}$$

Thus, since $\alpha\mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_1|^2\big] = \int_{\mathbb{R}^N}|\mathbf{y}|^2\,M(d\mathbf{y})$, the desired equality follows. □

§ **4.2.3. Poisson Jump Processes.** Rather than attempting to construct more general Lévy processes directly, I will first construct their jump processes and then construct them out of their jumps. With this idea in mind, I say that $(t,\omega)\rightsquigarrow j(t,\,\cdot\,,\omega)$ is a **Poisson jump process** associated with $M\in\mathfrak{M}_\infty(\mathbb{R}^N)$ if, for each $\omega\in\Omega$, $t\rightsquigarrow j(t,\,\cdot\,,\omega)$ is a jump function, and for each $n\in\mathbb{Z}^+$ and collection $\{\Delta_1,\dots,\Delta_n\}\subseteq\mathcal{B}_{\mathbb{R}^N}$ satisfying $\mathbf{0}\notin\bigcup_{i=1}^n\overline{\Delta_i}$, $\big\{\{j(t,\Delta_i): t\ge 0\}: 1\le i\le n\big\}$ are independent, simple Poisson processes, the ith of which is run at rate $M(\Delta_i)$ for each $1\le i\le n$. By starting with simple functions and passing to limits, one can easily check that

$$(t,\omega)\in[0,\infty)\times\Omega\longmapsto\int\varphi(\mathbf{y})\,j(t,d\mathbf{y},\omega)\in[0,\infty]$$

is measurable for every Borel measurable function $\varphi : \mathbb{R}^N \longrightarrow [0,\infty]$. Therefore, if $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$ is a Borel measurable function, and, for $T > 0$,

$$\Omega(T) \equiv \left\{\omega : \int |F(\mathbf{y})|\, j(T, d\mathbf{y}, \omega) < \infty\right\},$$

then both the set $\Omega(T)$ and the function

$$(t,\omega) \in [0,T] \times \Omega(T) \rightsquigarrow \int F(\mathbf{y})\, j(t,\mathbf{y},\omega) \in \mathbb{R}^{N'}$$

are measurable. Note that if $|F(\mathbf{y})|$ vanishes for $\mathbf{y}$'s in a neighborhood of $\mathbf{0}$, then $\Omega(T) = \Omega$ for all $T > 0$.

My goal in this subsection is to prove the following existence result.

THEOREM 4.2.11. *For each $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$ there exists an associated Poisson jump process. (See § 9.2.2 for another approach.)*

PROOF: Set $A_0 = \mathbb{R}^N \setminus \overline{B(\mathbf{0},1)}$ and $A_k = \overline{B(\mathbf{0},2^{-k+1})} \setminus \overline{B(\mathbf{0},2^{-k})}$ for $k \in \mathbb{Z}^+$, and define $M_k(d\mathbf{y}) = \mathbf{1}_{A_k}(\mathbf{y})\, M(d\mathbf{y})$. Next, choose independent Poisson processes $\big\{\{\mathbf{Z}_k(t) : t \geq 0\} : k \in \mathbb{N}\big\}$ so that the kth one is associated with M_k, and set $j_k(t,\,\cdot\,,\omega) = j\big(t,\,\cdot\,,\mathbf{Z}_k(\,\cdot\,,\omega)\big)$. Without loss in generality, I may and will assume that $j_k(t, A_k\complement, \omega) = 0$ for all $(t,\omega) \in [0,\infty) \times \Omega$ and $k \in \mathbb{N}$. In addition, by Theorem 4.2.9, if $\mathbf{Z}^{(m)}(t) = \sum_{k=0}^m \mathbf{Z}_k(t)$, then we know that, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$j^{(m)}(t,\,\cdot\,,\omega) \equiv j\big(t,\,\cdot\,,\mathbf{Z}^{(m)}(\,\cdot\,,\omega)\big) = \sum_{k=0}^m j_k(t,\,\cdot\,,\omega), \quad t \geq 0.$$

Hence, I may and will assume that

$$t \rightsquigarrow j(t,\,\cdot\,,\omega) \equiv \sum_{k=1}^\infty j_k(t,\,\cdot\,,\omega)$$

is a jump function for all $\omega \in \Omega$. Finally, suppose that $\{\Delta_i : 1 \leq i \leq n\} \subseteq \mathcal{B}_{\mathbb{R}^N}$ are disjoint and that $\mathbf{0} \notin \bigcup_{i=1}^n \overline{\Delta_i}$. Choose $m \in \mathbb{N}$ so that $(\bigcup_1^n \Delta_i) \cap \overline{B(\mathbf{0},2^{-m})} = \emptyset$, and note that, $\mathbb{P}$-almost surely, $j(t,\Delta_i,\omega) = j^{(m)}(t,\Delta_i,\omega)$ for all $t \geq 0$ and $1 \leq i \leq n$. Hence, the required property is a consequence of the last part of Theorem 4.2.8. □

In preparation for the next section, I prove the following.

LEMMA 4.2.12. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$ be a Borel measurable function such that $F(\mathbf{0}) = \mathbf{0}$ and $\mathbf{0} \notin \overline{F^{-1}\big(\mathbb{R}^{N'} \setminus B(\mathbf{0},r)\big)}$ for any $r > 0$. For any $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$, $M^F \in \mathfrak{M}_\infty(\mathbb{R}^{N'})$. Moreover, if $\{j(t,\,\cdot\,) : t \geq 0\}$ is a Poisson jump process* *associated with M, then (cf. Lemma 4.2.6) $\{j^F(t,\,\cdot\,) : t \geq 0\}$ is a Poisson jump process associated with M^F. Finally, if $\mathbf{0} \notin \overline{F^{-1}(\mathbb{R}^{N'} \setminus \{\mathbf{0}\})}$ and*

$$\mathbf{Z}^F(t,\omega) \equiv \int \mathbf{y}\, j^F(t, d\mathbf{y}, \omega) = \int F(\mathbf{y})\, j(t, d\mathbf{y}, \omega),$$

then $M^F \in \mathfrak{M}_0(\mathbb{R}^{N'})$, $\{\mathbf{Z}^F(t) : t \geq 0\}$ is a Poisson process associated with M^F, and $j\big(t,\,\cdot\,, \mathbf{Z}^F(\,\cdot\,,\omega)\big) = j^F(t,\,\cdot\,,\omega)$.

Proof: To prove the first assertion, suppose that $\{\Delta_1, \dots, \Delta_n\}$ are disjoint, Borel subsets of $\mathbb{R}^{N'}$ such that $\mathbf{0} \notin \bigcup_{i=1}^n \overline{\Delta_i}$. Then $\{F^{-1}(\Delta_1), \dots, F^{-1}(\Delta_n)\}$ satisfy the same conditions as subsets of $\mathbb{R}^N$, and therefore, since $j^F(t, \Delta_i, \omega) = j\big(t, F^{-1}(\Delta_i), \omega\big)$, $\big\{\{j^F(t,\Delta_i) : t \geq 0\} : 1 \leq i \leq n\big\}$ has the required properties.

Turning to the second assertion, first note that $M^F \in \mathfrak{M}_0(\mathbb{R}^{N'})$ is a immediate consequence of $\mathbf{0} \notin \overline{F^{-1}(\mathbb{R}^{N'} \setminus \{\mathbf{0}\})}$ and that the equality $j\big(t,\,\cdot\,, \mathbf{Z}^F(\,\cdot\,,\omega)\big) = j^F(t,\,\cdot\,,\omega)$ is a trivial application of the final part of Theorem 4.1.8. To prove that $\{\mathbf{Z}^F(t) : t \geq 0\}$ is a Poisson process associated with M^F, use Theorem 4.2.8 to see that $\{j^F(t,\,\cdot\,) : t \geq 0\}$ has the same distribution as the jump process for a Poisson process $\{\mathbf{Z}(t) : t \geq 0\}$ associated with M^F. Hence, since $\mathbf{Z}(t) = \int \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z})$, $\{\mathbf{Z}^F(t) : t \geq 0\}$ has the same distribution as $\{\mathbf{Z}(t) : t \geq 0\}$. □

§ **4.2.4. Lévy Processes with Bounded Variation.** Although the contents of the previous section provide the machinery with which to construct a Lévy process for any μ with Fourier transform given by (4.2.1), for reasons made clear in the next lemma, I will treat the special case when $M \in \mathfrak{M}_1(\mathbb{R}^N)$ here and will deal with $M \in \mathfrak{M}_2(\mathbb{R}^N) \setminus \mathfrak{M}_1(\mathbb{R}^N)$ in the following subsection.

Lemma 4.2.13. *Let $\{j(t,\,\cdot\,) : t \geq 0\}$ be a Poisson jump process associated with $M \in \mathfrak{M}_2(\mathbb{R}^N)$, and set $V(t,\omega) = \int |\mathbf{y}|\, j(t, d\mathbf{y}, \omega)$. Then $V(t) < \infty$ almost surely or $V(t) = \infty$ almost surely for all $t > 0$ depending on whether M is or is not in $\mathfrak{M}_1(\mathbb{R}^N)$. (See Exercise 4.3.11 to see that the same conclusion holds for any $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$.)*

Proof: Since $\int_{|\mathbf{y}|>1} |\mathbf{y}|\, j(t, d\mathbf{y}, \omega) < \infty$ for all $(t,\omega) \in [0,\infty) \times \Omega$, the question is entirely about the finiteness of $V_0(t,\omega) \equiv \int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|\, j(t, d\mathbf{y}, \omega)$. To study this question, set $A_k = \overline{B(\mathbf{0}, 2^{-k+1})} \setminus \overline{B(\mathbf{0}, 2^{-k})}$, $F_k(\mathbf{y}) = |\mathbf{y}|\mathbf{1}_{A_k}(\mathbf{y})$, and $V_k(t,\omega) = \int_{A_k} |\mathbf{y}|\, j(t, d\mathbf{y}, \omega)$ for $k \geq 1$. Clearly, the processes $\big\{\{V_k(t) : t \geq 0\} : k \in \mathbb{Z}^+\big\}$ are independent. In addition, $t \rightsquigarrow V_k(t)$ is non-decreasing and, by the second part of Lemma 4.2.12, $\{V_k(t) : t \geq 0\}$ is a Poisson process associated with M^{F_k}. Thus, by Lemma 4.2.10,

$$a_k \equiv \mathbb{E}^{\mathbb{P}}\big[V_k(t)\big] = t \int_{A_k} |\mathbf{y}|\, M(d\mathbf{y}) \text{ and } b_k \equiv \mathrm{Var}\big(V_k(t)\big) = t \int_{A_k} |\mathbf{y}|^2\, M(d\mathbf{y}).$$

From the first of these, it follows that

$$\mathbb{E}^{\mathbb{P}}\left[\int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|\, j(t, d\mathbf{y})\right] = \sum_{k=1}^{\infty} \mathbb{E}^{\mathbb{P}}\big[V_k(t)\big] = \int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|\, M(dy),$$

which finishes the case when $M \in \mathfrak{M}_1(\mathbb{R}^N)$. When $M \in \mathfrak{M}_2(\mathbb{R}^N) \setminus \mathfrak{M}_1(\mathbb{R}^N)$, set $\bar{V}_k(t) = V_k(t) - ta_k$. Then, for each $t > 0$, $\{\bar{V}_k(t) : k \in \mathbb{Z}^+\}$ is a sequence of independent random with mean value 0. Furthermore,

$$\sum_{k=1}^{\infty} \operatorname{Var}\big(\bar{V}_k(t)\big) = t \sum_{k=1}^{\infty} b_k = t \int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|^2\, M(d\mathbf{y}) < \infty.$$

Hence, by Theorem 1.4.2, $\sum_{k=1}^{\infty} \bar{V}_k(t)$ converges $\mathbb{P}$-almost surely. But, when $M \notin \mathfrak{M}_1(\mathbb{R}^N)$, $\sum_{k=1}^{\infty} a_k = \infty$, and so, for each $t > 0$, $\sum_{k=1}^{\infty} V_k(t)$ must diverge $\mathbb{P}$-almost surely. □

Before stating the main result of the subsection, I want to introduce the notion of a **generalized Poisson measure**. Namely, if $M \in \mathfrak{M}_1(\mathbb{R}^N) \setminus \mathfrak{M}_0(\mathbb{R}^N)$ and π_M is the element of $\mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is given by

$$\exp\left(\int \left(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1\right) M(d\mathbf{y})\right),$$

or, equivalently, $\widehat{\pi_M}$ is given by (4.2.1) with $\mathbf{m} = \int_{\overline{B(\mathbf{0},1)}} \mathbf{y}\, M(d\mathbf{y})$, then I will call π_M the generalized Poisson measure for M. Similarly, if $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for a generalized Poisson measure π_M, I will say that it is a **generalized Poisson process** associated with M.

Theorem 4.2.14. *Suppose that $M \in \mathfrak{M}_1(\mathbb{R}^N)$ and that $\{j(t, \cdot) : t \geq 0\}$ is a Poisson jump process associated with M. Set $\mathcal{N} = \{\omega : \exists t > 0\ j(t, \cdot, \omega) \notin \mathfrak{M}_1(\mathbb{R}^N)\}$, and define $(t,\omega) \rightsquigarrow \mathbf{Z}_M(t,\omega)$ so that*

$$\mathbf{Z}_M(t,\omega) = \begin{cases} \int \mathbf{y}\, j(t, d\mathbf{y}, \omega) & \text{if } \omega \notin \mathcal{N} \\ \mathbf{0} & \text{if } \omega \in \mathcal{N}. \end{cases}$$

Then $\mathbb{P}(\mathcal{N}) = 0$ and $\{\mathbf{Z}_M(t) : t \geq 0\}$ is a (possibly generalized) Poisson process associated with M. In particular, $t \rightsquigarrow \mathbf{Z}_M(t,\omega)$ is absolutely pure jump for all $\omega \in \Omega$, and $\{j(t, \cdot, \mathbf{Z}_M) : t \geq 0\}$ is a Poisson jump process associated with M. Finally, if $\mu \in \mathcal{I}(\mathbb{R}^N)$ has Fourier transform given by (4.2.1), then

$$\left\{ t\left(\mathbf{m} - \int_{\overline{B(\mathbf{0},1)}} \mathbf{y}\, M(d\mathbf{y})\right) + \mathbf{Z}_M(t) : t \geq 0 \right\}$$

is a Lévy process for μ.

PROOF: That $\mathbb{P}(\mathcal{N}) = 0$ follows from Lemma 4.2.13. To prove that $\{\mathbf{Z}_M(t) : t \geq 0\}$ is a Lévy process for π_M, set

$$\mathbf{Z}^{(r)}(t,\omega) = \int_{|\mathbf{y}|>r} \mathbf{y}\, j(t, d\mathbf{y}, \omega)$$

for $r > 0$. By Lemma 4.2.12, $\{\mathbf{Z}^{(r)}(t) : t \geq 0\}$ is a Poisson process associated with $M^{(r)}(d\mathbf{y}) \equiv \mathbf{1}_{(r,\infty)}(\mathbf{y})\, M(d\mathbf{y})$. In addition, if $\omega \notin \mathcal{N}$, then $\mathbf{Z}^{(r)}(\,\cdot\,,\omega) \longrightarrow \mathbf{Z}_M(\,\cdot\,,\omega)$ uniformly on compacts, from which it is easy to check that $\{\mathbf{Z}_M(t) : t \geq 0\}$ is a Poisson process associated with M and that the process in the last assertion is a Lévy process for the μ whose Fourier transform is given by (4.2.1) with this M. Finally, by the last part of Theorem 4.1.8, $j\big(t,\,\cdot\,,\mathbf{Z}_M(\,\cdot\,,\omega)\big) = j(t,\,\cdot\,,\omega)$ when $\omega \notin \mathcal{N}$, from which it is clear that $\{j(t,\,\cdot\,,\mathbf{Z}_M) : t \geq 0\}$ is a Poisson jump process associated with M. □

§ **4.2.5. General, Non-Gaussian Lévy Processes.** In this subsection I will complete the construction of non-Gaussian Lévy processes.

THEOREM 4.2.15. *For each $\mathbf{m} \in \mathbb{R}^N$ and $M \in \mathfrak{M}_2(\mathbb{R}^N)$ there is a Lévy process for the $\mu \in \mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is given by (4.2.1). Moreover, if $\{\mathbf{Z}(t) : t \geq 0\}$ is such a process, then $\{j(t,\,\cdot\,,\mathbf{Z}) : t \geq 0\}$ is a Poisson jump process associated with M. Finally, if, for $r \in (0,1]$,*

$$\mathbf{Z}^{(r)}(t) = \int_{|\mathbf{y}|>r} \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z}) - t\int_{r<|\mathbf{y}|\leq 1} \mathbf{y}\, M(d\mathbf{y}),$$

then

$$\mathbb{P}\left(\sup_{\tau\in[0,t]}\big|\mathbf{Z}(\tau) - \tau\mathbf{m} - \mathbf{Z}^{(r)}(\tau)\big| \geq \epsilon\right) \leq \frac{Nt}{\epsilon^2}\int_{\overline{B(\mathbf{0},r)}} |\mathbf{y}|^2\, M(d\mathbf{y}).$$

PROOF: Without loss in generality, I will assume that $\mathbf{m} = \mathbf{0}$.

By Theorem 4.2.11, we know that there is a Poisson jump process $\{j(t,\,\cdot\,) : t \geq 0\}$ associated with M. Take

$$\bar{j}(t, d\mathbf{y}, \omega) = j(t, d\mathbf{y}, \omega) - t\mathbf{1}_{\overline{B(\mathbf{0},1)}}(\mathbf{y})M(d\mathbf{y}),$$

and define

$$\mathbf{Z}^{(r)}(t,\omega) = \int_{|\mathbf{y}|>r} \mathbf{y}\, \bar{j}(t, d\mathbf{y}, \omega), \quad (t,\omega) \in [0,\infty)\times\Omega,$$

for $r \in (0,1]$. By Theorem 4.2.14, we know that $\{\mathbf{Z}^{(r)}(t) : t \geq 0\}$ is a Lévy process for $\mu^{(r)}$, where

$$\widehat{\mu^{(r)}}(\boldsymbol{\xi}) = \exp\left(\int_{|\mathbf{y}|>r}\Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\,\mathbf{1}_{[0,1]}(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y})\right).$$

Furthermore, by the second part of Lemma 4.2.10, we know that, for $0 < r < r' \le 1$,

$$(*) \qquad \mathbb{P}\big(\|\mathbf{Z}^{(r')} - \mathbf{Z}^{(r)}\|_{[0,t]} \ge \epsilon\big) \le \frac{Nt}{\epsilon^2} \int_{r<|\mathbf{y}|\le r'} |\mathbf{y}|^2 \, M(d\mathbf{y}).$$

Hence, if $1 \ge r_m \searrow 0$ is chosen so that

$$\int_{\overline{B(\mathbf{0},r_m)}} |\mathbf{y}|^2 \, M(d\mathbf{y}) \le 2^{-m},$$

then

$$\begin{aligned}\mathbb{P}\left(\sup_{n>m} \|\mathbf{Z}^{(r_n)} - \mathbf{Z}^{(r_m)}\|_{[0,t]} \ge \tfrac{1}{m}\right) &\le \sum_{n\ge m} \mathbb{P}\big(\|\mathbf{Z}^{(r_{n+1})} - \mathbf{Z}^{(r_n)}\|_{[0,t]} \ge (m+1)^{-2}\big) \\ &\le Nt \sum_{n=m}^{\infty} (n+1)^4 2^{-n},\end{aligned}$$

and therefore, by the first part of the Borel–Cantelli Lemma,

$$\mathbb{P}\left(\exists m \; \forall n \ge m \; \|\mathbf{Z}^{(r_n)} - \mathbf{Z}^{(r_m)}\|_{[0,t]} \le \tfrac{1}{m+1}\right) = 1.$$

We now know that there is a $\mathbb{P}$-null set $\mathcal{N}$ such that, for any $\omega \notin \mathcal{N}$, there exists a $\mathbf{Z}(\,\cdot\,,\omega) \in D(\mathbb{R}^N)$ to which $\{\mathbf{Z}^{(r_m)}(\,\cdot\,,\omega) : n \ge 0\}$ converges uniformly on compacts. Thus, if we take $\mathbf{Z}(t,\omega) = \mathbf{0}$ for $(t,\omega) \in [0,\infty) \times \mathcal{N}$, then is an easy matter to check that $\{\mathbf{Z}(t) : t \ge 0\}$ is a Lévy process for the $\mu \in \mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is given by (4.2.1) with $\mathbf{m} = \mathbf{0}$. In addition, since, by Theorem 4.1.8, we know that $t \rightsquigarrow j(t,\,\cdot\,,\omega)$ is the jump function for $t \rightsquigarrow \mathbf{Z}(t,\omega)$ when $\omega \notin \mathcal{N}$, it is clear that $\{j(t,\,\cdot\,,\mathbf{Z}) : t \ge 0\}$ is a Poisson jump process associated with M Finally, to prove the estimate in the concluding assertion, observe that, for $\omega \notin \mathcal{N}$, the path $t \rightsquigarrow \mathbf{Z}^{(r)}(t,\omega)$ used in our construction coincides with the path described in the statement. Thus, the desired estimate is an easy consequence of the one in (*) above. □

Corollary 4.2.16. *Let $\mu \in \mathcal{I}(\mathbb{R}^N)$ with Fourier transform given by (4.2.1), and suppose that $\{\mathbf{Z}(t) : t \ge 0\}$ is a Lévy process for μ. Then, depending on whether or not $M \in \mathfrak{M}_1(\mathbb{R}^N)$, either $\mathbb{P}$-almost all or $\mathbb{P}$-almost none of the paths $t \rightsquigarrow \mathbf{Z}(t)$ has locally bounded variation. Moreover, if $M \in \mathfrak{M}_1(\mathbb{R}^N)$, then, $\mathbb{P}$-almost surely,*

$$t \rightsquigarrow \mathbf{Z}(t) - t\left(\mathbf{m} - \int_{\overline{B(\mathbf{0},1)}} \mathbf{y} \, M(d\mathbf{y})\right) \quad \textit{is an absolutely pure jump path.}$$

Proof: From Theorem 4.2.14, we already know that $t \rightsquigarrow \mathbf{Z}(t) - t\mathbf{m}$ is almost surely an absolutely pure jump path if $M \in \mathfrak{M}_1(\mathbb{R}^N)$, and so $t \rightsquigarrow \mathbf{Z}(t)$ is almost surely of locally bounded variation. Conversely, if $t \rightsquigarrow \mathbf{Z}(t)$ has locally bounded variation with positive probability, then, by (4.1.13), $j\big(t,\,\cdot\,,\mathbf{Z}\big) \in \mathfrak{M}_1(\mathbb{R}^N)$ with positive probability. But then, since $\{j\big(t,\,\cdot\,,\mathbf{Z}\big) : t \ge 0\}$ is a Poisson jump process associated with M, it follows from Lemma 4.2.13 that $M \in \mathfrak{M}_1(\mathbb{R}^N)$. □

Corollary 4.2.17. *Let μ and $\{\mathbf{Z}(t) : t \ge 0\}$ be as in Corollary 4.2.16. Given $\Delta \in \mathcal{B}_{\mathbb{R}^N}$ with $\mathbf{0} \notin \bar{\Delta}$, set*

$$\mathbf{Z}^{\Delta}(t) = \int_{\Delta} \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z}),\; M^{\Delta}(d\mathbf{y}) = \mathbf{1}_{\Delta}(\mathbf{y}) M(d\mathbf{y}),\; \text{and } \mathbf{m}^{\Delta} = \int_{B(\mathbf{0},1)} \mathbf{y}\, M^{\Delta}(d\mathbf{y}).$$

Then $\{\mathbf{Z}^{\Delta}(t) : t \ge 0\}$ is a Poisson process associated with M^{Δ}, $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \ge 0\}$ is a Lévy process for the element of $\mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is

$$\exp\Bigg(\sqrt{-1}\big(\boldsymbol{\xi}, \mathbf{m} - \mathbf{m}^{\Delta}\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N \setminus \Delta} \Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y})\Bigg),$$

and $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \ge 0\}$ is independent of $\{j\big(t, \cdot, \mathbf{Z}^{\Delta}\big) : t \ge 0\}$, and therefore of $\{\mathbf{Z}^{\Delta}(t) : t \ge 0\}$ as well.

Proof: That $\{\mathbf{Z}^{\Delta}(t) : t \ge 0\}$ is a Poisson process associated with M^{Δ} is an immediate consequence of Lemma 4.2.12. Next, define $\mathbf{Z}^{(r)}(t)$ as in Theorem 4.2.15. Then, for all $r \in (0, 1]$,

$$\mathbf{Z}^{(r)}(t) - \mathbf{Z}^{\Delta}(t) = \int_{|\mathbf{y}|>r} \mathbf{1}_{\mathbb{R}^N \setminus \Delta}(\mathbf{y})\mathbf{y}\, j(t, d\mathbf{y}) - t \int_{r<|\mathbf{y}|\le 1} \mathbf{y}\, M(d\mathbf{y}).$$

In particular, this means that $\{\mathbf{Z}^{(r)}(t) - \mathbf{Z}^{\Delta}(t) : t \ge 0\}$ has independent, homogeneous increments and (cf. Theorem 4.1.8) is independent of $\{j\big(t, \cdot, \mathbf{Z}^{\Delta}\big) : t \ge 0\}$. Thus, since, as $r \searrow 0$, $\mathbf{Z}^{(r)}(t) \longrightarrow \mathbf{Z}(t) - t\mathbf{m}$ in probability, it follows that $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \ge 0\}$ is independent of $\{j(t, \cdot, \mathbf{Z}^{\Delta}) : t \ge 0\}$. In addition,

$$\begin{aligned} &e^{-\sqrt{-1}t(\boldsymbol{\xi}, \mathbf{m}-\mathbf{m}^{\Delta})_{\mathbb{R}^N}}\, \mathbb{E}^{\mathbb{P}}\big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t))_{\mathbb{R}^N}}\big] = \lim_{r \searrow 0} \mathbb{E}^{\mathbb{P}}\big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{Z}^{(r)}(t) - \mathbf{Z}^{\Delta}(t) + t\mathbf{m}^{\Delta})_{\mathbb{R}^N}}\big] \\ &= \lim_{r \searrow 0} \exp\left(\int_{(\Delta \cup \overline{B(\mathbf{0},r)})\complement} \Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y})\right) \\ &= \exp\left(\int_{\mathbb{R}^N \setminus \Delta} \Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y})\right). \end{aligned}$$

Hence, it follows that $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \ge 0\}$ is a Lévy process for the specified element of $\mathcal{I}(\mathbb{R}^N)$. $\square$

Exercises for § 4.2

Exercise 4.2.18. Here is another proof that the process $\{N(t) : t \geq 0\}$ in § 4.2.1 has independent, homogeneous increments. Refer to the notation used there.

(**i**) Given $n \in \mathbb{Z}^+$ and measurable functions $f : [0,\infty)^{n+1} \longmapsto [0,\infty)$ and $g : [0,\infty)^n \longrightarrow \mathbb{R}$, show that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\big[f(\tau_1,\dots,\tau_{n+1}),\, \tau_{n+1} > g(\tau_1,\dots,\tau_n)\big]\\ &\quad = \mathbb{E}^{\mathbb{P}}\big[e^{-g(\tau_1,\dots,\tau_n)^+} f\big(\tau_1,\dots,\tau_n,\tau_{n+1}+g(\tau_1,\dots,\tau_n)^+\big)\big].\end{aligned}$$

(**ii**) Let $K \in \mathbb{Z}^+$, $0 = n_0 \leq n_1 \leq \cdots \leq n_K$, and $0 = t_0 \leq t_1 < \cdots < t_K = s$ be given, and set $A = \{N(t_k) = n_k,\, 1 \leq k \leq K\}$. Show that $A = B \cap \{\tau_{n_K+1} > s - T_{n_K}\}$, where $B \in \sigma\big(\{\tau_1,\dots,\tau_{n_K}\}\big)$, and apply (**i**) to see that $\mathbb{P}(A) = \mathbb{E}^{\mathbb{P}}\big[e^{(s-T_{n_K})},\, B\big]$.

(**iii**) Let $n \in \mathbb{Z}^+$ and $t > 0$ be given, and set $h(\xi) = \mathbb{P}(T_{n-1} > \xi)$. Referring to (**ii**) and again using (**i**), show that

$$\begin{aligned}&\mathbb{P}\big(A \cap \{N(s+t) - N(s) < n\}\big) = \mathbb{E}^{\mathbb{P}}\big[h(t+s-T_{n_K+1}),\, B \cap \{\tau_{n_K+1} > s - T_{n_K}\}\big]\\ &= \mathbb{E}^{\mathbb{P}}\big[e^{-(s-T_{n_K})} h(t-\tau_{n_K+1}),\, B\big] = \mathbb{E}^{\mathbb{P}}\big[h(t-\tau_{n_K+1})\big]\mathbb{E}^{\mathbb{P}}\big[e^{-(s-T_{n_K})},\, B\big]\\ &= \mathbb{P}\big(N(t) < n\big)\mathbb{P}(A).\end{aligned}$$

∎

Exercise 4.2.19. Let $\{N(t) : t \geq 0\}$ be a simple Poisson process, and show that $\lim_{t\to\infty} \frac{N(t)}{t} = 1$ $\mathbb{P}$-almost surely.

Hint: First use the Strong Law of Large Numbers to show that $\lim_{n\to\infty} \frac{N(n)}{n} = 1$ $\mathbb{P}$-almost surely. Second, use

$$\mathbb{P}\left(\sup_{n \leq t \leq n+1} \frac{N(t) - N(n)}{t} \geq \epsilon\right) \leq \mathbb{P}\big(N(1) \geq n\epsilon\big) \leq \frac{2}{\epsilon^2 n^2},$$

to see that

$$\lim_{t\to\infty} \left| \frac{N(t)}{t} - \frac{N([t])}{[t]} \right| = 0 \quad \mathbb{P}\text{-almost surely.}$$

Exercise 4.2.20. Assume that $\mu \in \mathcal{I}(\mathbb{R})$ has its Fourier transform given by (4.2.1), and let $\{Z(t) : t \geq 0\}$ be a Lévy process for μ. Using Exercise 3.2.35, show that $t \rightsquigarrow Z(t)$ is non-decreasing if and only if $M \in \mathfrak{M}_1(\mathbb{R})$, $M\big((-\infty,0)\big) = 0$, and $m \geq \int_{[-1,1]} y\, M(dy)$.

Exercise 4.2.21. Let $\{j(t,\,\cdot\,) : t \geq 0\}$ be a Poisson jump process associated with some $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$, and suppose that $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Borel measurable, M-integrable function which vanishes at $\mathbf{0}$.

(**i**) Let $\mathcal{N}$ be the set of $\omega \in \Omega$ for which there is a $t > 0$ such that F is not integrable $j\big(t, \cdot, \omega\big)$-integrable, and show that $\mathbb{P}(\mathcal{N}) = 0$.

(**ii**) Show that (cf. Lemma 4.2.6) $M^F \in \mathfrak{M}_1(\mathbb{R})$ and that, in fact,

$$\int |\mathbf{y}|\, M^F(d\mathbf{y}) = \int |F(\mathbf{y})|\, M(d\mathbf{y}) < \infty.$$

Next, define

$$Z^F(t,\omega) = \begin{cases} \int F(\mathbf{y})\, j(t, d\mathbf{y}, \omega) & \text{if } \omega \notin \mathcal{N} \\ 0 & \text{if } \omega \in \mathcal{N}, \end{cases}$$

and show that $\{Z^F(t) : t \geq 0\}$ is a (possibly generalized) Poisson process associated with M^F.

(**iii**) Show that

$$\lim_{t\to\infty} \frac{Z^F(t)}{t} = \int F(\mathbf{y})\, M(d\mathbf{y}) \quad \mathbb{P}\text{-almost surely.}$$

Hint: Begin by using Lemma 4.2.10 to show that it suffices to handle F's which vanish in a neighborhood of $\mathbf{0}$. When F vanishes in a neighborhood of $\mathbf{0}$, use Lemma 4.2.12 to see that $\{Z^F(t) : t \geq 0\}$ is a Poisson process associated with M^F. Finally, use the representation of a Poisson process in terms of a simple Poisson process and independent random variables, and apply the Strong Law of Large Numbers together with the result in Exercise 4.2.19.

EXERCISE 4.2.22. Let $\{\mathbf{Z}(t) : t \geq 0\}$ be a Lévy process for the $\mu \in \mathcal{I}(\mathbb{R}^N)$ with Fourier transform given by (4.2.1), and set $\bar{\mathbf{Z}}(t) = \mathbf{Z}(t) - t\mathbf{m}$. Show that for all $R \in [1,\infty)$ and $t \in (0,\infty)$, $\mathbb{P}\big(\|\bar{\mathbf{Z}}\|_{[0,t]} \geq R\big)$ is dominated by t times

$$\frac{4N}{R^2}\int_{B(\mathbf{0},1)} |\mathbf{y}|^2\, M(d\mathbf{y}) + \frac{2}{R}\int_{1<|\mathbf{y}|\leq\sqrt{R}} |\mathbf{y}|\, M(d\mathbf{y}) + M\Big(\overline{B(\mathbf{0},\sqrt{R})}\complement\Big).$$

Hint: Write $\bar{\mathbf{Z}}(t) = \mathbf{Z}_1(t) + \mathbf{Z}_2(t) + \mathbf{Z}_3(t)$, where

$$\mathbf{Z}_2(t) = \int_{1<|\mathbf{y}|\leq\sqrt{R}} \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z}) \quad \text{and} \quad \mathbf{Z}_3(t) = \int_{|\mathbf{y}|>\sqrt{R}} \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z}).$$

Then,

$$\mathbb{P}\big(\|\mathbf{Z}\|_{[0,t]} \geq R\big) \leq \mathbb{P}\big(\|\mathbf{Z}_1\|_{[0,t]} \geq \tfrac{R}{2}\big) + \mathbb{P}\big(\|\mathbf{Z}_2\|_{[0,t]} \geq \tfrac{R}{2}\big) + \mathbb{P}\big(\|\mathbf{Z}_3\|_{[0,t]} \neq 0\big).$$

Apply the estimates in Lemma 4.2.10 to control the first two terms on the right, and use

$$\mathbb{P}\Big(j\big(t, \mathbb{R}^N \setminus \overline{B(\mathbf{0},\sqrt{R})}, \mathbf{Z}\big) \neq 0\Big) = 1 - e^{-tM(\mathbb{R}^N\setminus\overline{B(\mathbf{0},\sqrt{R})})}$$

to control the third.

EXERCISE 4.2.23. Let ν be a locally finite, Borel measure on $\mathbb{R}^N$. A **Poisson point process** with intensity measure ν is a random, locally finite, purely atomic measure-valued random variable $\omega \rightsquigarrow P(\,\cdot\,,\omega)$ with the properties that, for any bounded $\Gamma \in \mathcal{B}_{\mathbb{R}^N}$, $P(\Gamma)$ is a Poisson random variable with mean value $\nu(\Gamma)$ and, for any $n \geq 2$ and family $\{\Gamma_1, \dots, \Gamma_n\}$ of mutually disjoint, bounded, Borel subsets of $\mathbb{R}^N$, $\{P(\Gamma_1), \dots, P(\Gamma_n)\}$ are independent. The purpose of this exercise is to show how one can always construct such a Poisson point process.

(**i**) Define $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ so that $F(\mathbf{0}) = \mathbf{0}$ and $F(\mathbf{y}) = \frac{\mathbf{y}}{|\mathbf{y}|^2}$ for $\mathbf{y} \neq \mathbf{0}$. Clearly, F is 1 to 1 and onto, and both F and F^{-1} are Borel measurable. Assuming that $\nu(\{\mathbf{0}\}) = 0$, show that $M \equiv F_*\nu \in \mathfrak{M}_\infty(\mathbb{R}^N)$ and that $\nu = (F^{-1})_*M$.

(**ii**) Continue to assume that $\nu(\{\mathbf{0}\}) = 0$, let $\{j(t,\,\cdot\,) : t \geq 0\}$ be a Poisson jump process associated with the M in (**i**), and set $P(\,\cdot\,,\omega) = (F^{-1})_* j(1,\,\cdot\,,\omega)$. Show $\omega \rightsquigarrow P(\,\cdot\,,\omega)$ is a Poisson point process with intensity ν.

(**vi**) In order to handle ν's which charge $\mathbf{0}$, suppose $\nu(\{\mathbf{0}\}) > 0$. Choose a point $\mathbf{x} \in \mathbb{R}^N$ for which $\nu(\{\mathbf{x}\}) = 0$, define $\nu'(\Gamma) = \nu(\mathbf{x} + \Gamma)$, note that $\nu'(\{\mathbf{0}\}) = 0$, and construct a Poisson point process $\omega \rightsquigarrow P'(\,\cdot\,,\omega)$ with intensity measure ν'. Finally, define $P(\Gamma,\omega) = P'(\Gamma - \mathbf{x},\omega)$, and check that $\omega \rightsquigarrow P(\,\cdot\,,\omega)$ is a Poisson point process with intensity measure ν.

EXERCISE 4.2.24. Let $M \in \mathfrak{M}_2(\mathbb{R}^N)$ be given, and assume that exists a decreasing sequence $\{r_n : n \geq 0\} \subseteq (0,1]$ with $r_n \searrow 0$ such that

$$\mathbf{m} = \lim_{n\to\infty} \int_{r_n < |\mathbf{y}| \leq 1} \mathbf{y}\, M(d\mathbf{y})$$

exists. Let $\mu \in \mathcal{I}(\mathbb{R}^N)$ have Fourier transform given by (4.2.1) with this $\mathbf{m}$ and M. If $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for μ, set

$$\mathbf{Z}_n(t,\omega) = \int_{|\mathbf{y}| > r_n} \mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}(\,\cdot\,,\omega)\big),$$

and show that $\lim_{n\to\infty} \mathbb{P}\big(\|\mathbf{Z} - \mathbf{Z}_n\|_{[0,t]} \geq \epsilon\big) = 0$ for all $t \geq 0$ and $\epsilon > 0$. Thus, after passing to a subsequence $\{n_m : m \geq 0\}$ if necessary, one sees that, $\mathbb{P}$-almost surely,

$$\mathbf{Z}(t,\omega) = \lim_{m\to\infty} \int_{|\mathbf{y}| > r_{n_m}} \mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}(\,\cdot\,,\omega)\big),$$

where the convergence is uniform on finite time intervals. In particular, one can say that $\mathbb{P}$-almost all the paths $t \rightsquigarrow \mathbf{Z}(t,\omega)$ are "conditionally pure jump."

§ 4.3 Brownian Motion, the Gaussian Lévy Process

What remains of the program in this chapter is the construction of a Lévy process for the standard, normal distribution $\gamma_{\mathbf{0},\mathbf{I}}$, the infinitely divisible law whose Fourier transform is $e^{-\frac{|\boldsymbol{\xi}|^2}{2}}$. Indeed, if $\{\mathbf{Z}_{\gamma_{\mathbf{0},I}}(t) : t \ge 0\}$ is such a process and $\{\mathbf{Z}_\mu(t) : t \ge 0\}$ is a Lévy process for the $\mu \in \mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is given by (4.2.1), and if $\{\mathbf{Z}_{\gamma_{\mathbf{0},I}}(t) : t \ge 0\}$ is independent of $\{\mathbf{Z}_\mu(t) : t \ge 0\}$, then it is an easy matter to check that $\mathbf{C}^{\frac{1}{2}}\mathbf{Z}_{\gamma_{\mathbf{0},I}}(t) + \mathbf{Z}_\mu(t)$ will be a Lévy process for $\gamma_{\mathbf{0},I} \star \mu$, whose Fourier transform is

$$\exp\Bigg(\sqrt{-1}\big(\boldsymbol{\xi},\mathbf{m}\big)_{\mathbb{R}^N} - \tfrac{1}{2}\big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N}\Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\,\mathbf{1}_{[0,1]}(|\mathbf{y}|)\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big]\,M(d\mathbf{y})\Bigg).$$

Because one of its earliest applications was as a mathematical model for the motion of "Brownian particles" * such a Lévy process for $\gamma_{0,1}$ is called a **Brownian motion**. In recognition of its provenance, I will adopt this terminology and will use the notation $\{\mathbf{B}(t) : t \ge 0\}$ instead of $\{\mathbf{Z}_{\gamma_{\mathbf{0},I}}(t) : t \ge 0\}$.

Before getting into the details, it may be helpful to think a little about what sort of properties we should expect the paths $t \rightsquigarrow \mathbf{B}(t)$ will possess. For this purpose, set $M_n = n\big(\delta_{n^{-\frac{1}{2}}} + \delta_{-n^{-\frac{1}{2}}}\big)^N$, and recall that we have seen already that $\pi_{M_n} \Longrightarrow \gamma_{\mathbf{0},\mathbf{I}}$. Since a Poisson process associated with M_n has nothing but jumps of size $n^{-\frac{1}{2}}$, if one believes that the Lévy process for $\gamma_{\mathbf{0},\mathbf{I}}$ should be, in some sense, the limit of such Poisson processes, then it is reasonable to guess that its paths will have jumps of size 0. That is, they will be continuous.

Although the prediction that the paths of $\{\mathbf{B}(t) : t \ge 0\}$ will be continuous is correct, it turns out that, because it is based on the Central Limit Theorem, the heuristic reasoning just given does not lead to the easiest construction. The problem is that the Central Limit Theorem gives convergence of distributions, not random variables, and therefore one should not expect the paths, as opposed to their distributions, of the approximating Poisson processes to converge. For this reason, it is easier to avoid the Central Limit Theorem and work with Gaussian random variables from the start, and that is what I will do here. The Central Limit approach is the content of § 9.3.

§ **4.3.1. Deconstructing Brownian Motion.** My construction of Brownian motion is based on an idea of Lévy's; and in order to explain Lévy's idea, I will begin with the following line of reasoning.

Assume that $\{\mathbf{B}(t) : t \ge 0\}$ is a Brownian motion in $\mathbb{R}^N$. That is, $\{\mathbf{B}(t) : t \ge 0\}$ starts at $\mathbf{0}$, has independent increments, any increment $\mathbf{B}(s+t) - \mathbf{B}(s)$ has distribution $\gamma_{\mathbf{0},t\mathbf{I}} \in N(\mathbf{0}, t\mathbf{I})$, and the paths $t \rightsquigarrow \mathbf{B}(t)$ are continuous. Next,

* R. Brown, an 18th Century English botanist, observed the motion of pollen articles in a dilute gas. His observations were interpreted by A. Einstein as evidence for the kinetic theory of gases. In his famous 1905 paper, Einstein took the first steps in a program, eventually completed by N. Wiener in 1923, to give a mathematical model of what Brown had seen.

given $n \in \mathbb{N}$, let $t \rightsquigarrow \mathbf{B}_n(t)$ be the polygonal path obtained from $t \rightsquigarrow \mathbf{B}(t)$ by linear interpolation during each time interval $[m2^{-n}, (m+1)2^{-n}]$. Thus,

$$\mathbf{B}_n(t) = \mathbf{B}(m2^{-n}) + 2^n\big(t - m2^{-n}\big)\Big(\mathbf{B}\big((m+1)2^{-n}\big) - \mathbf{B}(m2^{-n})\Big)$$

for $m2^{-n} \le t \le (m+1)2^{-n}$. The distribution of $\{\mathbf{B}_0(t) : t \ge 0\}$ is very easy to understand. Namely, if $\mathbf{X}_{m,0} = \mathbf{B}(m) - \mathbf{B}(m-1)$ for $m \ge 1$, then the $\mathbf{X}_{m,0}$'s are independent, standard normal $\mathbb{R}^N$-valued random variable, $\mathbf{B}_0(m) = \sum_{1 \le m \le n} \mathbf{X}_{m,0}$, and $\mathbf{B}_0(t) = (m-t)\mathbf{B}_0(m-1) + (t-m+1)\mathbf{B}_0(m)$ for $m-1 \le t \le m$. To understand the relationship between successive $\mathbf{B}_n$'s, observe that $\mathbf{B}_{n+1}(m2^{-n}) = \mathbf{B}_n(m2^{-n})$ for all $m \in \mathbb{N}$ and that

$$\begin{aligned}
\mathbf{X}_{m,n+1} &\equiv 2^{\frac{n}{2}+1}\Big(\mathbf{B}_{n+1}\big((2m-1)2^{-n-1}\big) - \mathbf{B}_n\big((2m-1)2^{-n-1}\big)\Big) \\
&= 2^{\frac{n}{2}+1}\left(\mathbf{B}\big((2m-1)2^{-n-1}\big) - \frac{\mathbf{B}\big(m2^{-n}\big) + \mathbf{B}\big((m-1)2^{-n}\big)}{2}\right) \\
&= 2^{\frac{n}{2}}\Big[\Big(\mathbf{B}\big((2m-1)2^{-n-1}\big) - \mathbf{B}\big((m-1)2^{-n}\big)\Big) \\
&\qquad\qquad - \Big(\mathbf{B}\big(m2^{-n}\big) - \mathbf{B}\big((2m-1)2^{-n-1}\big)\Big)\Big],
\end{aligned}$$

and therefore $\{\mathbf{X}_{m,n+1} : m \ge 1\}$ is again a sequence of independent standard normal random variables. What is less obvious is that $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$ is also a family of independent random variables. In fact, checking this requires us to make essential use of the fact that we are dealing with Gaussian random variables.

In preparation for proving the preceding independence assertion, say that $\mathfrak{G} \subseteq L^2(\mathbb{P};\mathbb{R})$ is a **Gaussian family** if $\mathfrak{G}$ is a linear subspace and each element of $\mathfrak{G}$ is a **centered** (i.e., mean-value 0) Gaussian random variable. My immediate interest in Gaussian families at this point is that the linear span $\mathfrak{G}(\mathbf{B})$ of $\big\{\big(\boldsymbol{\xi}, \mathbf{B}(t)\big)_{\mathbb{R}^N} : t \ge 0 \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N\big\}$ is one. To see this, simply note that, for any $0 = t_0 < t_1 < \cdots t_n$ and $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n \in \mathbb{R}^N$,

$$\sum_{m=1}^n \big(\boldsymbol{\xi}_m, \mathbf{B}(t_m)\big)_{\mathbb{R}^N} = \sum_{\ell=1}^n \left(\sum_{m=\ell}^n \big(\boldsymbol{\xi}_m, \mathbf{B}(t_\ell) - \mathbf{B}(t_{\ell-1})\big)_{\mathbb{R}^N}\right)_{\mathbb{R}^N},$$

which, as a linear combination of independent centered Gaussians, is itself a centered Gaussian.

The crucial fact about Gaussian families is the content of the next lemma.

LEMMA 4.3.1. *Suppose that $\mathfrak{G} \subseteq L^2(\mathbb{P};\mathbb{R})$ is a Gaussian family. Then the closure of $\mathfrak{G}$ in $L^2(\mathbb{P};\mathbb{R})$ is again a Gaussian family. Moreover, for any $S \subseteq \mathfrak{G}$, S is independent of $S^\perp \cap \mathfrak{G}$, where $S^\perp$ is the orthogonal complement of S in $L^2(\mathbb{P};\mathbb{R})$.*

Proof: The first assertion is easy since, as I noted in the introduction to Chapter III, Gaussian random variables are closed under convergence in probability.

Turning to the second part, what we must show is that if $X_1, \dots, X_n \in S$ and $X'_1, \dots, X'_n \in S^\perp \cap \mathfrak{G}$, then (cf. part (**ii**) of Exercise 1.1.13)

$$\mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi_m X_m} \prod_{m=1}^n e^{\sqrt{-1}\,\xi'_m X'_m}\right] = \mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi_m X_m}\right]\mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi'_m X'_m}\right]$$

for any choice of $\{\xi_m : 1 \le m \le n\} \cup \{\xi'_m : 1 \le m \le n\} \subseteq \mathbb{R}$. But the expectation value on the left is equal to

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=1}^n \big(\xi_m X_m + \xi'_m X'_m\big)\right)^2\right]\right)\\
&\quad = \exp\left(-\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=1}^n \xi_m X_m\right)^2\right] - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=1}^n \xi'_m X'_m\right)^2\right]\right)\\
&\quad = \mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi_m X_m}\right]\mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi'_m X'_m}\right],
\end{aligned}$$

since $\mathbb{E}^{\mathbb{P}}[X_m X'_{m'}] = 0$ for all $1 \le m, m' \le n$. □

Armed with Lemma 4.3.1, we can now check that $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$ is independent. Indeed, since, for all $(m,n) \in \mathbb{Z}^+ \times \mathbb{N}$ and $\boldsymbol{\xi} \in \mathbb{R}^N$, $\big(\boldsymbol{\xi}, \mathbf{X}_{m,n}\big)_{\mathbb{R}^N}$ a member of the Gaussian family $\mathfrak{G}(\mathbf{B})$, all that we have to do is check that, for each $(m,n) \in \mathbb{Z}^+ \times \mathbb{N}$, $\ell \in \mathbb{N}$, and $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in (\mathbb{R}^N)^2$,

$$\mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{X}_{m,n+1}\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\big)_{\mathbb{R}^N}\big] = 0.$$

But, since, for $s \le t$, $\mathbf{B}(s)$ is independent of $\mathbf{B}(t) - \mathbf{B}(s)$,

$$\mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{B}(s)\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(t)\big)_{\mathbb{R}^N}\big] = \mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{B}(s)\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(s)\big)_{\mathbb{R}^N}\big] = s\big(\boldsymbol{\xi}, \boldsymbol{\eta}\big)_{\mathbb{R}^N}$$

and therefore

$$\begin{aligned}
&2^{-\frac{n}{2}-1}\mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{X}_{m,n+1}\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\big)_{\mathbb{R}^N}\big]\\
&\quad = \mathbb{E}^{\mathbb{P}}\Big[\Big(\boldsymbol{\xi}, \mathbf{B}\big((2m-1)2^{-n-1}\big)\Big)_{\mathbb{R}^N}\Big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\Big)_{\mathbb{R}^N}\Big]\\
&\qquad\qquad - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\Big[\Big(\boldsymbol{\xi}, \mathbf{B}\big(m2^{-n}\big) + \mathbf{B}\big((m-1)2^{-n}\big)\Big)_{\mathbb{R}^N}\Big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\Big)_{\mathbb{R}^N}\Big]\\
&\quad = 2^{-n}\big(\boldsymbol{\xi}, \boldsymbol{\eta}\big)_{\mathbb{R}^N}\left[\big(m - \tfrac{1}{2}\big) \wedge \ell - \frac{m \wedge \ell + (m-1) \wedge \ell}{2}\right] = 0.
\end{aligned}$$

§ **4.3.2. Lévy's Construction of Brownian Motion.** Lévy's idea was to invert the reasoning given in the preceding subsection. That is, start with a family $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$ of independent $N(\mathbf{0},\mathbf{I})$-random variables. Next, define $\{\mathbf{B}_n(t) : t \geq 0\}$ inductively so that $t \rightsquigarrow \mathbf{B}_n(t)$ is linear on each interval $[(m-1)2^{-n}, m2^{-n}]$, $\mathbf{B}_0(m) = \sum_{1\leq \ell \leq m} \mathbf{X}_{\ell,0}$, $m \in \mathbb{N}$, $\mathbf{B}_{n+1}(m2^{-n}) = \mathbf{B}_n(m2^{-n-1})$ for $m \in \mathbb{N}$, and

$$\mathbf{B}_{n+1}\big((2m-1)2^{-n}\big) = \mathbf{B}_n\big((2m-1)2^{-n-1}\big) + 2^{-\frac{n}{2}-1}\mathbf{X}_{m,n+1} \quad \text{for } m \in \mathbb{Z}^+.$$

If Brownian motion exists, then the distribution of $\{\mathbf{B}_n(t) : t \geq 0\}$ is the distribution of the process obtained by polygonalizing it on each of the intervals $[(m-1)2^{-n}, m2^{-n}]$, and so the limit $\lim_{n\to\infty} \mathbf{B}_n(t)$ should exist uniformly on compacts and should be Brownian motion.

To see that this procedure works, one must first verify that the preceding definition of $\{\mathbf{B}_n(t) : t \geq 0\}$ gives a process with the correct distribution. That is, we need to show that $\big\{\mathbf{B}_n\big((m+1)2^{-n}\big) - \mathbf{B}_n\big(m2^{-n}\big) : m \in \mathbb{N}\big\}$ is a sequence of independent $N(\mathbf{0}, 2^{-n}\mathbf{I})$-random variables. But, since this sequence is contained in the Gaussian family spanned by $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$, Lemma 4.3.1 says that we will know this once we show that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[&\Big(\boldsymbol{\xi}, \mathbf{B}_n\big((m+1)2^{-n}\big) - \mathbf{B}_n\big(m2^{-n}\big)\Big)_{\mathbb{R}^N} \\ &\times \Big(\boldsymbol{\xi}', \mathbf{B}_n\big((m'+1)2^{-n}\big) - \mathbf{B}_n\big(m'2^{-n}\big)\Big)_{\mathbb{R}^N}\Big] = 2^{-n}\big(\boldsymbol{\xi}, \boldsymbol{\xi}'\big)_{\mathbb{R}^N}\delta_{m,m'}\end{aligned}$$

for $\boldsymbol{\xi}$, $\boldsymbol{\xi}' \in \mathbb{R}^N$ and m, $m' \in \mathbb{N}$. When $n = 0$, this is obvious. Now assume that it is true for n, and observe that

$$\begin{aligned}&\mathbf{B}_{n+1}(m2^{-n}) - \mathbf{B}_{n+1}\big((2m-1)2^{-n-1}\big) \\ &\qquad = \frac{\mathbf{B}_n(m2^{-n}) - \mathbf{B}_n\big((m-1)2^{-n}\big)}{2} - 2^{-\frac{n}{2}-1}\mathbf{X}_{m,n+1}\end{aligned}$$

and

$$\begin{aligned}&\mathbf{B}_{n+1}\big((2m-1)2^{-n-1}\big) - \mathbf{B}_{n+1}\big((m-1)2^{-n}\big) \\ &\qquad = \frac{\mathbf{B}_n(m2^{-n}) - \mathbf{B}_n\big((m-1)2^{-n}\big)}{2} + 2^{-\frac{n}{2}-1}\mathbf{X}_{m,n+1}.\end{aligned}$$

Using these expressions and the induction hypothesis, it is easy to check the required equation.

Second, and more challenging, we must show that, $\mathbb{P}$-almost surely, these processes are converging uniformly on compact time intervals. For this purpose, consider the difference $t \rightsquigarrow \mathbf{B}_{n+1}(t) - \mathbf{B}_n(t)$. Since this path is linear on each interval $[m2^{-n-1}, (m+1)2^{-n-1}]$,

$$\max_{t\in[0,2^L]} \big|\mathbf{B}_{n+1}(t) - \mathbf{B}_n(t)\big| = \max_{1\le m\le 2^{L+n+1}} \big|\mathbf{B}_{n+1}(m2^{-n-1}) - \mathbf{B}_n(m2^{-n-1})\big|$$
$$= 2^{-\frac{n}{2}-1} \max_{1\le m\le 2^{L+n+1}} |\mathbf{X}_{m,n+1}| \le 2^{-\frac{n}{2}-1} \left(\sum_{m=1}^{2^{L+n+1}} |\mathbf{X}_{m,n+1}|^4 \right)^{\frac{1}{4}}.$$

Thus, by Jensen's inequality,

$$\mathbb{E}^{\mathbb{P}}\big[\|\mathbf{B}_{n+1} - \mathbf{B}_n\|_{[0,2^L]}\big] \le 2^{-\frac{n}{2}-1} \left(\sum_{m=1}^{2^{L+n+1}} \mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_{m,n+1}|^4\big] \right)^{\frac{1}{4}} = 2^{-\frac{n-L+3}{4}} C_N$$

where $C_N \equiv \mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_{1,0}|^4\big]^{\frac{1}{4}} < \infty$.

Starting from the preceding, it is an easy matter to show that there is a measurable $\mathbf{B} : [0,\infty) \times \Omega \longrightarrow \mathbb{R}^N$ such that $\mathbf{B}(0) = \mathbf{0}$, $\mathbf{B}(\,\cdot\,,\omega) \in C\big([0,\infty);\mathbb{R}^N\big)$ for each $\omega \in \Omega$, and $\|\mathbf{B}_n - \mathbf{B}\|_{[0,t]} \longrightarrow 0$ in $\mathbb{P}$-almost surely and in $L^1(\mathbb{P};\mathbb{R})$ for every $t \in [0,\infty)$. Furthermore, since $\mathbf{B}(m2^{-n}) = \mathbf{B}_n(m2^{-n})$ $\mathbb{P}$-almost surely for all $(m,n) \in \mathbb{N}^2$, it is clear that $\big\{\mathbf{B}\big((m+1)2^{-n}\big) - \mathbf{B}(m2^{-n}) : m \ge 0\big\}$ is a sequence of independent $N(\mathbf{0}, 2^{-n}\mathbf{I})$-random variables for all $n \in \mathbb{N}$. Hence, by continuity, it follows that $\{\mathbf{B}(t) : t \ge 0\}$ is a Brownian motion.

We have now completed the task described in the introduction to this section. However, before moving on, it is only proper to recognize that, clever as his method is, Lévy was not the first to construct a Brownian motion. Instead, it was N. Wiener who was the first. In fact, his famous* 1923 article "Differential Space" in *J. Math. Phys.* #**2** contains three different approaches.

§ **4.3.3. Lévy's Construction in Context.** There are elements of Lévy's construction which admit interesting generalizations, perhaps the most important of which is **Kolmogorov's Continuity Criterion**.

THEOREM 4.3.2. *Suppose that $\{X(t) : t \in [0,T]\}$ is a family of random variables taking values in a Banach space B, and assume that, for some $p \in [1,\infty)$, $C < \infty$, and $r \in (0,1]$,*

$$\mathbb{E}^{\mathbb{P}}\big[\|X(t) - X(s)\|_B^p\big]^{\frac{1}{p}} \le C|t-s|^{\frac{1}{p}+r} \quad \textit{for all } s,\, t \in [0,T].$$

Then, there exists a family $\{\tilde{X}(t) : t \in [0,T]\}$ such that $X(t) = \tilde{X}(t)$ $\mathbb{P}$-almost surely for each $t \in [0,T]$ and $t \in [0,T] \longmapsto \tilde{X}(t,\omega) \in B$ is continuous for all

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* Wiener's article is remarkable, but I must admit that I have never been convinced that it is complete. Undoubtedly, my doubts are more a consequence of my own ineptitude than of his.

$\omega \in \Omega$. In fact, for each $\alpha \in (0, r)$,

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\le s<t\le T}\frac{\|\tilde{X}(t)-\tilde{X}(s)\|_B}{(t-s)^\alpha}\right] \le \frac{5CT^{\frac{1}{p}+r-\alpha}}{(1-2^{-r})(1-2^{\alpha-r})}.$$

PROOF: First note that, by rescaling time, it suffices to treat the case when $T=1$.

Given $n \ge 0$, set $M_n = \max_{1\le m\le 2^n}\big\|X(m2^{-n}) - X\big((m-1)2^{-n}\big)\big\|_B$, and observe that

$$\mathbb{E}^{\mathbb{P}}\big[M_n\big] \le \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=1}^{2^n}\big\|X(m2^{-n}) - X\big((m-1)2^{-n}\big)\big\|_B^p\right)^{\frac{1}{p}}\right] \le C2^{-rn}.$$

Next, let $t \rightsquigarrow X_n(t)$ be the polygonal path obtained by linearizing $t \rightsquigarrow X(t)$ on each interval $[(m-1)2^{-n}, m2^{-n}]$, and check that

$$\begin{aligned}&\max_{t\in[0,1]}\|X_{n+1}(t)-X_n(t)\|_B\\ &\quad= \max_{1\le m\le 2^n}\left\|X\big((2m-1)2^{-n-1}\big) - \frac{X\big((m-1)2^{-n}\big)-X(m2^{-n})}{2}\right\|_B \le M_{n+1}.\end{aligned}$$

Hence, $\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,1]}\|X_{n+1}(t)-X_n(t)\|_B\Big] \le C2^{-rn}$, and so there exists a measurable $\tilde{X} : [0,1]\times\Omega \longrightarrow B$ such that $t \rightsquigarrow \tilde{X}(t,\omega)$ is continuous for all $\omega \in \Omega$ and

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,1]}\|\tilde{X}(t)-X_n(t)\|_B\right] \le \frac{C2^{-rn}}{1-2^{-r}}.$$

Moreover, because, for each $t \in [0,1]$, $\|X(\tau)-X(t)\|_B \longrightarrow 0$ in probability as $\tau \to t$, it is easy to check that, for each $t\in[0,1]$, $\tilde{X}(t) = X(t)$ $\mathbb{P}$-almost surely.

To prove the final estimate, note that for $2^{-n-1} \le t-s \le 2^{-n}$ one has that

$$\begin{aligned}\|\tilde{X}(t)-\tilde{X}(s)\|_B &\le \|\tilde{X}(t)-X_n(t)\|_B + \|X_n(t)-X_n(s)\|_B + \|X_n(s)-\tilde{X}(s)\|_B\\ &\le 2\sup_{\tau\in[0,1]}\|\tilde{X}(\tau)-X_n(\tau)\|_B + 2^n(t-s)M_n,\end{aligned}$$

and therefore that

$$\frac{\|\tilde{X}(t)-\tilde{X}(s)\|_B}{(t-s)^\alpha} \le 2 2^{\alpha(n+1)}\sup_{\tau\in[0,1]}\|\tilde{X}(\tau)-X_n(\tau)\|_B + 2^n 2^{(\alpha-1)n}M_n.$$

But, by the estimates proved above, this means that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{0\le s<t\le 1}\frac{\|\tilde{X}(t)-\tilde{X}(s)\|_B}{(t-s)^\alpha}\right] &\le C\sum_{n=0}^{\infty}\left(2\frac{2^{\alpha(n+1)}2^{-rn}}{1-2^{-r}} + 2^{\alpha n}2^{-rn}\right)\\ &\le \frac{5C}{(1-2^{-r})(1-2^{\alpha-r})}. \quad \square\end{aligned}$$

Corollary 4.3.3. *If $\{\mathbf{B}(t) : t \geq 0\}$ is an $\mathbb{R}^N$-valued Brownian motion, then, for each $\alpha \in \left(0, \frac{1}{2}\right)$, $t \rightsquigarrow \mathbf{B}(t)$ is $\mathbb{P}$-almost surely Hölder continuous of order α. In fact, for each $T \in (0,\infty)$,*

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq s<t\leq T} \frac{|\mathbf{B}(t)-\mathbf{B}(s)|}{(t-s)^\alpha}\right] < \infty.$$

Proof: In view of Theorem 4.3.2, all that we have to do is note that, for each $n \in \mathbb{Z}^+$, there is a $C_n < \infty$ such that $\mathbb{E}^{\mathbb{P}}\big[|\mathbf{B}(t)-\mathbf{B}(s)|^{2n}\big] \leq C_n|t-s|^n$. □

§ **4.3.4. Brownian Paths are Non-differentiable.** Having shown that Brownian paths are Hölder continuous of every order strictly less than $\frac{1}{2}$, I will close this section by showing that they are nowhere Hölder continuous of any order strictly greater than $\frac{1}{2}$. In particular, this will prove Wiener's famous result that *Brownian paths are nowhere differentiable*. The proof which follows is due to A. Devoretzky.

Theorem 4.3.4. *Let $\{\mathbf{B}(t) : t \geq 0\}$ be an $\mathbb{R}^N$-valued Brownian motion. Then, for each $\alpha > \frac{1}{2}$,*

$$\mathbb{P}\left(\exists s \in [0,\infty)\ \varlimsup_{t\searrow s} \frac{|\mathbf{B}(t)-\mathbf{B}(s)|}{(t-s)^\alpha} < \infty\right) = 0.$$

Proof: Because $\{\mathbf{B}(T+t)-\mathbf{B}(T) : t \geq 0\}$ is a Brownian motion for each $T \in [0,\infty)$, it suffices for us to show that

$$\mathbb{P}\left(\exists s \in [0,1)\ \varlimsup_{t\searrow s} \frac{|\mathbf{B}(t)-\mathbf{B}(s)|}{(t-s)^\alpha} < \infty\right) = 0.$$

To this end, note that, for every $L \in \mathbb{Z}^+$,

$$\begin{aligned}&\left\{\exists s \in [0,1)\ \varlimsup_{t\searrow s} \frac{|\mathbf{B}(t)-\mathbf{B}(s)|}{(t-s)^\alpha} < \infty\right\}\\ &\qquad \subseteq \bigcup_{M=1}^\infty \bigcup_{\nu=1}^\infty \bigcap_{n=\nu}^\infty \bigcup_{m=0}^n \bigcap_{\ell=0}^{L-1} \left\{\left|\mathbf{B}\left(\tfrac{m+\ell+1}{n}\right)-\mathbf{B}\left(\tfrac{m+\ell}{n}\right)\right| \leq \tfrac{M}{n^\alpha}\right\}.\end{aligned}$$

Thus, it enough to show that there is a choice of L such that

$$\lim_{n\to\infty} n\mathbb{P}\Big(\left|\mathbf{B}\left(\tfrac{\ell+1}{n}\right)-\mathbf{B}\left(\tfrac{\ell}{n}\right)\right| \leq \tfrac{M}{n^\alpha},\ 0\leq \ell < L\Big) = 0.$$

But

$$\begin{aligned}&\mathbb{P}\Big(\left|\mathbf{B}\left(\tfrac{\ell+1}{n}\right)-\mathbf{B}\left(\tfrac{\ell}{n}\right)\right| \leq \tfrac{M}{n^\alpha},\ 0\leq \ell < L\Big)\\ &\quad = \gamma_{\mathbf{0},\frac{1}{n}\mathbf{I}}\Big(\overline{B\big(\mathbf{0},\tfrac{M}{n^\alpha}\big)}\Big)^L = \left((2\pi)^{-\frac{N}{2}}\int_{B(\mathbf{0},Mn^{\frac{1}{2}-\alpha})} e^{-\frac{|\mathbf{y}|^2}{2}}\,d\mathbf{y}\right)^L \leq Cn^{(\frac{1}{2}-\alpha)NL}.\end{aligned}$$

Hence, we need only take L so that $(\alpha - \frac{1}{2})NL > 1$. $\square$

In spite of their being non-differentiable, "differentials" of Brownian paths display remarkable regularity properties. To wit, I make the following simple observation. In its statement, $\|\cdot\|_{\mathrm{H.S.}}$ denotes the Hilbert–Schmidt norm on $\mathrm{Hom}(\mathbb{R}^N;\mathbb{R}^N)$.

THEOREM 4.3.5. *If $\{\mathbf{B}(t) : t \geq 0\}$ is an $\mathbb{R}^N$-valued Brownian motion, then, for each $T \in (0,\infty)$*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| \sum_{m=1}^{[nt]} \big(\Delta_{m,n}\mathbf{B}\big) \otimes \big(\Delta_{m,n}\mathbf{B}\big) - t\mathbf{I} \right\|_{\mathrm{H.S.}} = 0 \quad \mathbb{P}\text{-almost surely,}$$

where $\Delta_{m,n}\mathbf{B} \equiv \mathbf{B}\left(\frac{m}{n}\right) - \mathbf{B}\left(\frac{m-1}{n}\right)$. In particular, $\mathbb{P}$-almost no Brownian path has locally bounded variation.

PROOF: Let $(\mathbf{e}_1,\dots,\mathbf{e}_N)$ be an orthonormal basis for $\mathbb{R}^N$, and set $X_i(k,n) = \big(\mathbf{e}_i, \Delta_{k,n}\mathbf{B}\big)_{\mathbb{R}^N}$. Then, what we have to show is that

$$(*) \qquad \lim_{n\to\infty} \sup_{1\leq m\leq nT} \left| \sum_{k=1}^{m} X_i(k,n)X_j(k,n) - \frac{m}{n}\delta_{i,j} \right| = 0 \quad \mathbb{P}\text{-almost surely.}$$

To this end, note that, for each $n \in \mathbb{Z}^+$ and $1 \leq i, j \leq N$, $\{X_i(k,n) : k \geq 1 \;\&\; 1 \leq i \leq N\}$ are independent $N(0,n^{-1})$-random variables. Hence, for each $1 \leq i \leq N$, $\{X_i(k,n)^2 - n^{-1} : k \geq 1\}$ are independent random variables with mean value 0 and variance $2n^{-2}$, and therefore, by (1.4.22) and the second inequality in (1.3.2),

$$\begin{aligned}
&\mathbb{E}\left[\left| \max_{1\leq m\leq nT} \sum_{k=1}^{m} \Big(X_i(k,n)^2 - \tfrac{1}{n}\Big) \right|^4 \right] \\
&\qquad \leq 4\mathbb{E}\left[\left| \sum_{1\leq k\leq nT} \Big(X_i(k,n)^2 - \tfrac{1}{n}\Big) \right|^4 \right] \leq \frac{12M_4T^2}{n^2},
\end{aligned}$$

where M_4 fourth moment of $X_1(1,1)^2 - 1$, and so the Borel–Cantelli Lemma can be used to check (*) when $i = j$. When $i \neq j$, the argument is essentially the same, only, because $X_i(k,n)X_j(k,n)$ has mean value 0, there is no need to subtract off its mean.

To prove the final assertion, note that if $\psi \in C\big([0,T];\mathbb{R}\big)$ has bounded variation, then

$$\lim_{n\to\infty} \sum_{m=1}^{[nT]} \Big(\psi\big(\tfrac{m}{n}\big) - \psi\big(\tfrac{m-1}{n}\big)\Big)^2 = 0. \quad \square$$

§ **4.3.5. General Lévy Processes.** Our original reason for constructing Brownian motion was to complete the program of constructing all the Lévy processes. In this subsection, I will do that.

Throughout this subsection, $\mu \in \mathcal{I}(\mathbb{R}^N)$ has Fourier transform

$$\tag{4.3.6} \exp\biggl(\sqrt{-1}\,(\boldsymbol{\xi},\mathbf{m})_{\mathbb{R}^N} - \tfrac{1}{2}(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi})_{\mathbb{R}^N} + \int\Bigl[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}(|\mathbf{y}|)(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\Bigr]\,M(d\mathbf{y})\biggr),$$

where $\mathbf{m} \in \mathbb{R}^N$, $\mathbf{C} \in \operatorname{Hom}(\mathbb{R}^N;\mathbb{R}^N)$ is symmetric and non-negative definite, and $M \in \mathfrak{M}_2(\mathbb{R}^N)$. In addition, I will use μ_0 to denote $\gamma_{\mathbf{m},\mathbf{C}}$ and μ_1 to denote the element of $\mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is

$$\exp\biggl(\int\Bigl[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}(|\mathbf{y}|)(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\Bigr]\,M(d\mathbf{y})\biggr).$$

Thus, $\mu = \mu_0 \star \mu_1$.

THEOREM 4.3.7. *There is a Lévy process $\{\mathbf{Z}(t) : t \ge 0\}$ for each $\mu \in \mathcal{I}(\mathbb{R}^N)$. Furthermore, if μ_0 and μ_1 are as the preceding discussion and if $\{\mathbf{Z}(t) : t \ge 0\}$ is a Lévy process for $\mu = \mu_0 \star \mu_1$, then there exist independent Lévy processes $\{\mathbf{Z}_0(t) : t \ge 0\}$ and $\{\mathbf{Z}_1(t) : t \ge 0\}$ for μ_0 and μ_1, respectively, such that $\mathbf{Z}(t) = \mathbf{Z}_0(t) + \mathbf{Z}_1(t)$, $t \ge 0$, $\mathbb{P}$-almost surely. In fact, if, for $r \in (0,1]$,*

$$\mathbf{Z}^{(r)}(t) = \int_{|\mathbf{y}|>r} \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z}) - t\int_{r<|\mathbf{y}|\le 1} \mathbf{y}\, M(d\mathbf{y}),$$

then, for each $t \in (0,\infty)$,

$$\mathbb{P}\bigl(\|\mathbf{Z}^{(r)} - \mathbf{Z}_1\|_{[0,t]} \ge \epsilon\bigr) \le \frac{Nt}{\epsilon^2}\int_{\overline{B(\mathbf{0},r)}} |\mathbf{y}|^2\, M(d\mathbf{y}).$$

PROOF: Let $\{\mathbf{B}(t) : t \ge 0\}$ be a Brownian motion and $\{\mathbf{Z}_1(t) : t \ge 0\}$ an independent Lévy process for μ_1, and define $\mathbf{Z}_0(t) = t\mathbf{m} + \mathbf{C}^{\frac{1}{2}}\mathbf{B}(t)$ and $\mathbf{Z}(t) = \mathbf{Z}_0(t)+\mathbf{Z}_1(t)$. As I pointed out in the introduction to this section, $\{\mathbf{Z}_0(t) : t \ge 0\}$ is a Lévy process for μ_0 and $\{\mathbf{Z}(t) : t \ge 0\}$ is a Lévy process for μ. Furthermore, because $t \rightsquigarrow \mathbf{Z}_0(t)$ is continuous, $j(t, \cdot\,, \mathbf{Z}) = j(t, \cdot\,, \mathbf{Z}_1)$. Hence, by the last part of Theorem 4.2.15, we know that the last part of the present theorem holds for this choice of $\{\mathbf{Z}(t) : t \ge 0\}$. Finally, since every Lévy process for μ will have the same distribution as this one, there is nothing more to do. $\square$

COROLLARY 4.3.8. *Let $\{\mathbf{Z}(t) : t \ge 0\}$ be a Lévy process for μ. Then $t \rightsquigarrow \mathbf{Z}(t)$ is $\mathbb{P}$-almost surely continuous if and only if $M = 0$ and is $\mathbb{P}$-almost surely of locally bounded variation if and only if $\mathbf{C} = 0$ and $M \in \mathfrak{M}_1(\mathbb{R}^N)$. Finally, $t \rightsquigarrow \mathbf{Z}(t)$ is $\mathbb{P}$-almost surely an absolutely pure jump path if and only if $\mathbf{C} = 0$, $M \in \mathfrak{M}_1(\mathbb{R}^N)$, and $\mathbf{m} = \int_{\overline{B(0,1)}} \mathbf{y}\, M(d\mathbf{y})$.*

PROOF: Let $\mathbf{Z}(t) = \mathbf{Z}_0(t) + \mathbf{Z}_1(t)$ be the decomposition described in Theorem 4.3.7, and let $\{j(t,\cdot\,) : t \ge 0\}$ be the jump process for $\{\mathbf{Z}(t) : t \ge 0\}$. If $M = 0$, then $\mathbf{Z}_1(t) = \mathbf{0}$, $t \ge 0$, $\mathbb{P}$-almost surely, and so $t \rightsquigarrow \mathbf{Z}(t) = \mathbf{Z}_0(t)$ is continuous $\mathbb{P}$-almost surely. Conversely, if $t \rightsquigarrow \mathbf{Z}(t)$ is continuous $\mathbb{P}$-almost surely, then $j(t,\cdot\,) = 0$, $t \ge 0$, $\mathbb{P}$-almost surely. Hence, since $\{j(t,\cdot\,) : t \ge 0\}$ is a Poisson jump process associated with M, we see that $M = 0$. Next, suppose that $\mathbf{C} = 0$. Then $\mathbf{Z}(t) = \mathbf{Z}_1(t) + t\mathbf{m}$, $t \ge 0$, $\mathbb{P}$-almost surely and therefore, by Corollary 4.2.16, $t \rightsquigarrow \mathbf{Z}(t)$ has locally bounded variation $\mathbb{P}$-almost surely if and only if $M \in \mathfrak{M}_1(\mathbb{R}^N)$ and is $\mathbb{P}$-almost surely an absolutely pure jump path if and only if $M \in \mathfrak{M}_1(\mathbb{R}^N)$ and $\mathbf{m} = \int_{\overline{B(\mathbf{0},1)}} \mathbf{y}\, M(d\mathbf{y})$. Thus, all that remains is to show that $\mathbf{C} = 0$ if $t \rightsquigarrow \mathbf{Z}(t)$ $\mathbb{P}$-almost surely has locally bounded variation. But, if $t \rightsquigarrow \mathbf{Z}(t)$ has locally bounded variation $\mathbb{P}$-almost surely, then, by (4.1.13), $\int |\mathbf{y}| j(t, d\mathbf{y}) < \infty$, $t \ge 0$, $\mathbb{P}$-almost surely and therefore, by Lemma 4.2.13, $M \in \mathfrak{M}_1(\mathbb{R}^N)$, which, by Corollary 4.2.16, implies that $t \rightsquigarrow \mathbf{Z}_1(t)$ has locally bounded variation $\mathbb{P}$-almost surely. Since this means that $t \rightsquigarrow \mathbf{Z}_0(t)$ must also have locally bounded variation $\mathbb{P}$-almost surely, and, since $\{\mathbf{Z}_0(t) : t \ge 0\}$ has the same distribution as $\{t\mathbf{m} + \mathbf{C}^{\frac{1}{2}}\mathbf{B}(t) : t \ge 0\}$, Theorem 4.3.5 shows that this is possible only if $\mathbf{C} = 0$. □

REMARK 4.3.9. Recall the linear functional A_μ introduced in (3.2.10). As I showed in Lemma 3.2.15, the action of A_μ on φ decomposes into a local part and a non-local part, which, with 20-20 hindsight, we can write as, respectively,

$$\begin{gathered}\big(\mathbf{m}, \nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N} + \tfrac{1}{2}\mathrm{Trace}\big(\mathbf{C}\nabla^2\varphi(\mathbf{0})\big)\\ \text{and } \int \Big[\varphi(\mathbf{y}) - \varphi(\mathbf{0}) - \mathbf{1}_{[0,1]}(|\mathbf{y}|)\big(\mathbf{y}, \nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y}).\end{gathered}$$

In terms of this decomposition, Corollary 4.3.8 is saying that the local part of A_μ governs the continuous part of $\{\mathbf{Z}(t) : t \ge 0\}$ and that the non-local part governs the discontinuous part.

Exercises for § 4.3

EXERCISE 4.3.10. This exercise deals with a few elementary facts about Brownian motion.

(**i**) Let $\{\mathbf{X}(t) : t \ge 0\}$ be an $\mathbb{R}^N$-valued stochastic process satisfying $\mathbf{X}(0,\omega) = \mathbf{0}$ and $\mathbf{X}(\,\cdot\,,\omega) \in C(\mathbb{R}^N)$ for all $\omega \in \Omega$, and show that $\{\mathbf{X}(t) : t \ge 0\}$ is an $\mathbb{R}^N$-valued Brownian motion if and only if the span of $\big\{\big(\boldsymbol{\xi}, \mathbf{X}(t)\big)_{\mathbb{R}^N} : t \ge 0 \;\&\; \boldsymbol{\xi} \in \mathbb{R}^N\big\}$ is a Gaussian family with the property that, for all t, $t' \in [0,\infty)$ and $\boldsymbol{\xi}$, $\boldsymbol{\xi}' \in \mathbb{R}^N$,

$$\mathbb{E}^{\mathbb{P}}\Big[\big(\boldsymbol{\xi}, \mathbf{X}(t)\big)_{\mathbb{R}^N}\big(\boldsymbol{\xi}', \mathbf{X}(t')\big)_{\mathbb{R}^N}\Big] = t \wedge t'(\boldsymbol{\xi}, \boldsymbol{\xi}')_{\mathbb{R}^N}.$$

(**ii**) Assuming that $\{\mathbf{B}(t) : t \ge 0\}$ is an $\mathbb{R}^N$-valued Brownian motion, show that $\{\mathcal{O}\mathbf{B}(t) : t \ge 0\}$ is also an $\mathbb{R}^N$-valued Brownian motion for any orthogonal transformation $\mathcal{O}$. That is, *the distribution of Brownian motion is invariant under rotation.* (See Theorem 8.3.14 for a significant generalization.)

(**iii**) Assuming that $\{\mathbf{B}(t) : t \geq 0\}$ is an $\mathbb{R}^N$-valued Brownian motion, show that $\{\lambda^{-\frac{1}{2}}\mathbf{B}(\lambda t) : t \geq 0\}$ is also an $\mathbb{R}^N$-Brownian motion for each $\lambda \in (0,\infty)$. This is called the *Brownian scaling invariance* property.

EXERCISE 4.3.11. This exercise introduces the *time inversion invariance* property of Brownian motion.

(**i**) Suppose that $\{\mathbf{B}(t) : t \geq 0\}$ is an $\mathbb{R}^N$-valued Brownian motion, and set $\mathbf{X}(t) = t\mathbf{B}\big(\frac{1}{t}\big)$ for $t > 0$. As an application of (**i**) in Exercise 4.3.10, show that $\{\mathbf{X}(t) : t > 0\}$ has the same distribution as $\{\mathbf{B}(t) : t > 0\}$, and conclude from this that $\lim_{t\searrow 0} \mathbf{X}(t) = \mathbf{0}$ $\mathbb{P}$-almost surely. In particular, if $\tilde{\mathbf{B}}(0,\omega) = \mathbf{0}$ and, for $t \in (0,\infty)$,

$$\tilde{\mathbf{B}}(t,\omega) = \begin{cases} t\mathbf{B}\left(\frac{1}{t},\omega\right) & \text{when } \lim_{\tau\to 0} \tau\mathbf{B}\left(\frac{1}{\tau},\omega\right) = \mathbf{0} \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

show that $\{\tilde{\mathbf{B}}(t) : t \geq 0\}$ is an $\mathbb{R}^N$-valued Brownian motion.

(**ii**) As a consequence of part (**i**), prove the *Brownian Strong Law of Large Numbers*: $\lim_{t\to\infty} t^{-1}\mathbf{B}(t) = \mathbf{0}$.

EXERCISE 4.3.12. Let $\{\mathbf{B}(t) : t \geq 0\}$ be an $\mathbb{R}^N$-valued Brownian motion.

(**i**) As an application of Theorem 1.4.13, show that, for any $\mathbf{e} \in \mathbb{S}^{N-1}$ and $T \in (0,\infty)$,

$$\mathbb{P}\left(\sup_{t\in[0,T]} \big|\big(\mathbf{e},\mathbf{B}(t)\big)\big| \geq R\right) \leq 2\mathbb{P}\Big(\big|\big(\mathbf{e},\mathbf{B}(T)\big)_{\mathbb{R}^N}\big| \geq R\Big) \leq 2e^{-\frac{R^2}{2T}},$$

and conclude that

$$\mathbb{P}\big(\|\mathbf{B}\|_{[0,T]} \geq R\big) \leq 2Ne^{-\frac{R^2}{2NT}}. \tag{4.3.13}$$

(**ii**) Now assume that $N = 1$, and set $B^*(t) = \max_{\tau\in[0,t]} B(\tau)$. Just as in part (**i**), use Theorem 1.4.13 to show that $\mathbb{P}\big(B^*(1) \geq a\big) \leq 2\mathbb{P}\big(B(1) \geq a\big)$ for all $a > 0$. By examining its proof, one sees that the inequality in Theorem 1.4.13 comes from not knowing how far over a the partial sums jumps when they first exceed level a. Thus, because we are now dealing with "continuous partial sums," one should suspect that the inequality can be made an equality. To verify this suspicion, let $\Gamma_n(\epsilon)$ denote the set of ω such that $|B(t,\omega) - B(s,\omega)| < \epsilon$ for all $0 \leq s < t \leq 1$ with $t - s \leq 2^{-n}$, and show that, for $0 < \epsilon < a$,

$$\begin{aligned} &\{B(1) \geq a\} \cap \Gamma_n(\epsilon) \\ &\qquad \subseteq \bigcup_{m=1}^{2^n-1} \left\{\max_{0\leq \ell<m} B(\ell 2^{-n}) < a - \epsilon \leq B(m2^{-n}) \;\&\; B(1) - B(m2^{-n}) > 0\right\}, \end{aligned}$$

and conclude that $\mathbb{P}\big(\{B(1) \geq a\} \cap \Gamma_n(\epsilon)\big) \leq \frac{1}{2}\mathbb{P}\big(B^*(1) \geq a - \epsilon\big)$ for all $n \in \mathbb{N}$. Now let $n \to \infty$ and then $\epsilon \searrow 0$ to arrive at $\mathbb{P}\big(B^*(1) \geq a\big) \geq 2\mathbb{P}\big(B(1) \geq a\big)$.

(**iii**) By combining the preceding with Brownian scaling invariance, arrive at

$$\mathbb{P}\big(B^*(t) \geq a\big) = 2\mathbb{P}\big(B(t) \geq a\big) = (2\pi)^{-\frac{1}{2}} \int_{at^{-\frac{1}{2}}}^{\infty} e^{-\frac{x^2}{2}}\, dx. \tag{4.3.14}$$

This beautiful result, which is sometimes called the **reflection principle for Brownian motion** seems to have appeared first in L. Bachelier's now famous 1900 thesis, where he used what is now called "Brownian motion" to model price fluctuations on the Paris Bourse. More information about the reflection principle can be found in § 8.6.3.

Exercise 4.3.15. Let $\{B(t) : t \geq 0\}$ be an $\mathbb{R}$-valued Brownian motion. The goal of this exercise is to prove the **Brownian law of the iterated logarithm**:

$$\varlimsup_{t\to\infty} \frac{B(t)}{\sqrt{2t \log_{(2)} t}} = 1 = \varlimsup_{t\searrow 0} \frac{B(t)}{\sqrt{2t \log_{(2)} t^{-1}}} \quad \mathbb{P}\text{-almost surely.}$$

Begin by checking that the second equality follows from the first applied to the time inverted process $\{\tilde{B}(t) : t \geq 0\}$ described in (**i**) of Exercise 4.3.11. Next, observe that

$$\varlimsup_{n\to\infty} \frac{B(n)}{\sqrt{2n \log_{(2)} n}} = 1 \quad \mathbb{P}\text{-almost surely}$$

is just the Law of the Iterated Logarithm for standard normal random variables. Thus, all that remains is to show that

$$\varlimsup_{n\to\infty} \sup_{t\in[n,n+1]} \left| \frac{B(t)}{\sqrt{2t \log_{(2)} t}} - \frac{B(n)}{\sqrt{2n \log_{(2)} 2n}} \right| = 0 \quad \mathbb{P}\text{-almost surely,}$$

which can be checked by a combination of the Strong Law for Brownian motion, the estimate in (4.3.13), and the easy half of the Borel–Cantelli Lemma.

Exercise 4.3.16. Given a stochastic process $\{X(t) : t \geq 0\}$, the stochastic process $\{\tilde{X}(t) : t \geq 0\}$ is said to be a **modification** of $\{X(t) : t \geq 0\}$ if, for each $t \in [0, \infty)$, $\tilde{X}(t) = X(t)$ $\mathbb{P}$-almost surely. Further, given a stochastic process $\{X(t) : t \geq 0\}$ with values in a metric space (E, ρ), one says that $\{X(t) : t \geq 0\}$ is **stochastically continuous** if, as $t \to s$, $X(t) \longrightarrow X(s)$ in probability for each $s \in [0, \infty)$.

(**i**) Show that the simple Poisson process $\{N(t) : t \geq 0\}$ is stochastically continuous. Thus, *stochastic continuity does not imply path continuity.*

(**ii**) Let $\mathbb{Q}$ denote the set of rational real numbers. Show that an $\mathbb{R}^N$-valued, stochastically continuous stochastic process $\{X(t) : t \ge 0\}$ admits a continuous modification if and only if, for each $T > 0$, $t \in [0,T] \cap \mathbb{Q} \longmapsto X(t)$ is uniformly continuous. Conclude that a stochastically continuous process $\{X(t) : t \ge 0\}$ admits a continuous modification if and only if there exists a $\mu \in \mathbf{M}_1\big(C(\mathbb{R}^N)\big)$ such that the distribution of $\{X(t) : t \ge 0\}$ under $\mathbb{P}$ is the same as the distribution of $\{\boldsymbol{\psi}(t) : t \ge 0\}$ under μ. Equivalently, a stochastically continuous process $\{X(t) : t \ge 0\}$ admits a continuous modification if and only if there exists a continuous stochastic process $\{Y(t) : t \ge 0\}$, not necessarily on the same probability space, with the same distribution as $\{X(t) : t \ge 0\}$.

Exercise 4.3.17. It is important to realize that the insistence in Theorem 4.3.2 that pth moment of $|X(t) - X(s)|$ be dominated by $|t-s|$ to a power strictly greater than p is essential. To see this, recall the simple Poisson process $\{N(t) : t \ge 0\}$ in § 5.2.1, and set $X(t) = N(t) - t$. The paths of this process are right-continuous but definitely not continuous. On the other hand, show that $\mathbb{E}^{\mathbb{P}}\big[\big(N(t) - N(s) - (t-s)\big)^2\big] \le t - s \quad \text{for } 0 \le s < t$. More generally, knowing that $\mathbb{E}\big[|X(t) - X(s)|^2\big]$ is dominated by $|t-s|$ is not enough to conclude that there is a continuous modification of $t \rightsquigarrow X(t)$.

Exercise 4.3.18. There is an important extension of Theorem 4.3.2 to processes which have a multi-dimensional parametrization. Let B be a Banach space and $\{X(\mathbf{x}) : \mathbf{x} \in [0,T]^\nu\}$ a family of B-valued random variables with the property that

$$\mathbb{E}^{\mathbb{P}}\big[\|X(\mathbf{y}) - X(\mathbf{x})\|_B^p\big]^{\frac{1}{p}} \le C|\mathbf{y} - \mathbf{x}|^{\frac{\nu}{p}+r}$$

for some $p \in [1,\infty)$, $r > 0$, and $C < \infty$. Show that there exists a family $\{\tilde{X}(\mathbf{x}) : \mathbf{x} \in [0,T]^\nu\}$ with the properties that $\mathbf{x} \in [0,T]^\nu \longmapsto \tilde{X}(\mathbf{x},\omega) \in B$ is continuous for all ω, and, for each $\mathbf{x} \in [0,T]^\nu$, $\tilde{X}(\mathbf{x},\omega) = X(\mathbf{x},\omega)$ $\mathbb{P}$-almost surely. Further, show that for each $\alpha \in (0,r)$, there is a universal $K(\nu,r,\alpha) < \infty$ such that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{\substack{\mathbf{x},\mathbf{y}\in[0,T]^\nu \\ \mathbf{y}\ne\mathbf{x}}} \frac{\|\tilde{X}(\mathbf{y}) - \tilde{X}(\mathbf{x})\|_B}{|\mathbf{y}-\mathbf{x}|^\alpha}\right] \le K(\nu,r,\alpha)CT^{\frac{\nu}{p}+r-\alpha}.$$

Hint: First rescale time to reduce to the case when $T = 1$. Now assume that $T = 1$. Given $n \in \mathbb{N}$, take S_n to be the set of pairs $(\mathbf{m},\mathbf{m}') \in \big(\{0,\dots,2^n\}^N\big)^2$ such that $m_i' \ge m_i$ for all $1 \le i \le \nu$ and $\sum_{i=1}^\nu (m_i' - m_i) = 1$, note that S_n has no more than $\nu 2^{(n+1)\nu}$ elements, set

$$M_n = \max\big\{\|X(\mathbf{m}'2^{-n}) - X(\mathbf{m}2^{-n})\|_B : (\mathbf{m},\mathbf{m}') \in S_n\big\},$$

and show that $\mathbb{E}^{\mathbb{P}}[M_n] \le C2^\nu \nu^{\frac{1}{p}} 2^{-rn}$. Next, let $\mathbf{x} \rightsquigarrow X_n(\mathbf{x})$ denote the nth dyadic multi-liniarization of $x \rightsquigarrow X(\mathbf{x})$, the one which is multilinear on each dyadic cube $\prod_{i=1}^{N}[(m_i-1)2^{-n}, m_i 2^{-n}]$ for $(m_1,\dots,m_N)\in\{1,\dots,2^n\}^N$. As in the proof of Theorem 4.3.2, argue that $\|X_{n+1}-X_n\|_{\mathrm{u},B}\le M_{n+1}$, and conclude that there exists an $(\mathbf{x},\omega)\rightsquigarrow \tilde{X}(\mathbf{x},\omega)$ which is continuous in $\mathbf{x}$ for each ω and is $\mathbb{P}$-almost surely equal to $X(\mathbf{x},\cdot\,)$ for each $\mathbf{x}$. Finally, to derive the Hölder continuity estimate, observe that $\|X_n(\mathbf{y})-X_n(\mathbf{x})\|_B\le 2^n\nu^{\frac{1}{2}}|\mathbf{y}-\mathbf{x}|M_n$, and proceed as in the proof of the corresponding part of Theorem 4.3.2.

Exercise 4.3.19. In this exercise we will examine a couple of the implications that Theorem 4.3.5 has about any Riemann–Stieltjes type integration theory involving Brownian paths. For simplicity, I will restrict my attention to the one dimensional case. Thus, let $\{B(t): t\ge 0\}$ be an $\mathbb{R}$-valued Brownian motion. Because $t\rightsquigarrow B(t)$ is continuous, one knows that any function $\psi:[0,1]\longrightarrow\mathbb{R}$ of bounded variation is Riemann–Stieltjes integrable on $[0,1]$ with respect to $B\restriction[0,1]$. However, as the following shows, almost no Brownian path is Riemann–Stieltjes with respect to itself. Namely, using Theorem 4.3.5, show that $\mathbb{P}$-almost surely,

$$\lim_{n\to\infty}\sum_{m=1}^{n}B\big(\tfrac{m-1}{n}\big)\Big(B\big(\tfrac{m}{n}\big)-B\big(\tfrac{m-1}{n}\big)\Big)=\frac{B(1)^2-1}{2},$$
$$\lim_{n\to\infty}\sum_{m=1}^{n}B\big(\tfrac{m}{n}\big)\Big(B\big(\tfrac{m}{n}\big)-B\big(\tfrac{m-1}{n}\big)\Big)=\frac{B(1)^2+1}{2},$$

whereas

$$\lim_{n\to\infty}\sum_{m=1}^{n}B\big(\tfrac{2m-1}{2n}\big)\Big(B\big(\tfrac{m}{n}\big)-B\big(\tfrac{m-1}{n}\big)\Big)=B(1)^2.$$

Exercise 4.3.20. Say that a $D(\mathbb{R}^N)$-valued process $\{\mathbf{Z}(t): t\ge 0\}$ is a Lévy process if $\mathbf{Z}(0)=\mathbf{0}$ and it has independent, homogeneous increments. Show that every Lévy process is a Lévy process for some $\mu\in\mathcal{I}(\mathbb{R}^N)$.

Exercise 4.3.21. Let $\{j(t,\cdot\,): t\ge 0\}$ be a Poisson jump process associated with some $M\in\mathfrak{M}_\infty(\mathbb{R}^N)$. In Lemma 4.2.13, we showed that when $M\in\mathfrak{M}_2(\mathbb{R}^N)$, then $\int|\mathbf{y}|\,j(t,d\mathbf{y})<\infty$, $t\ge 0$, with positive probability only if $M\in\mathfrak{M}_1(\mathbb{R}^N)$. In this exercise, we will show that the same is true for any $M\in\mathfrak{M}_\infty(\mathbb{R}^N)$. Thus, assume that $\int|\mathbf{y}|\,j(t,d\mathbf{y})<\infty$, $t\ge 0$, with positive probability, and show that $M\in\mathfrak{M}_1(\mathbb{R}^N)$.

(**i**) As an application of Kolmogorov's 0–1 Law, show that $\int|\mathbf{y}|\,j(t,d\mathbf{y})<\infty$ with positive probability implies it is finite with probability 1.

(**ii**) Let $\mathcal{N}$ be the set of $\omega\in\Omega$ for which there is a $t>0$ such that $\int|\mathbf{y}|\,j(t,d\mathbf{y},\omega)=\infty$. By (**i**), $\mathbb{P}(\mathcal{N})=0$. Define $\mathbf{Z}(t,\omega)=\int\mathbf{y}\,j(t,d\mathbf{y},\omega)$ for $\omega\notin\mathcal{N}$ and $\mathbf{Z}(t,\omega)=\mathbf{0}$ for $\omega\in\mathcal{N}$, and show that $\{\mathbf{Z}(t): t\ge 0\}$ is a Lévy process with absolutely pure jump paths.

(**iii**) Applying Theorem 4.1.8, first show that $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for a μ with Lévy measure M, and then apply Corollary 4.3.8 to conclude that $M \in \mathfrak{M}_1(\mathbb{R}^N)$.

EXERCISE 4.3.22. Corollary 4.3.3 can be sharpened. In fact, Lévy showed that if $\{B(t) : t \geq 0\}$ is an $\mathbb{R}$-valued Brownian motion, then

$$\mathbb{P}\left(\lim_{\delta \searrow 0} \sup_{0<t-s\leq\delta} \frac{|B(t)-B(s)|}{L(\delta)} = \sqrt{2}\right) = 1,$$

where $L(\delta) \equiv \sqrt{\delta \log \delta^{-1}}$. Notice that, on the one hand, this result is in the direction that one should expect: we know (cf. Theorem 4.3.4) that Brownian paths are almost never Hölder continuous of any order greater than $\frac{1}{2}$. On the other hand, the Brownian Law of the Iterated Logarithm (cf. Exercise 4.3.15) might make one guess that their true modulus of continuity ought to be $\sqrt{\delta \log_{(2)} \delta^{-1}}$, not $L(\delta)$. However, that guess is wrong because it fails to take into account the difference between a question about what is true at a single time as opposed to what is true simultaneously for all times. The purpose of this exercise is to show how the considerations in § 4.3.3 can be used to get a statement which is less refined than Lévy's. The result proved here shows only that

$$\mathbb{P}\left(\varlimsup_{\delta \searrow 0} \sup_{0<t-s\leq\delta} \frac{|B(t)-B(s)|}{L(\delta)} \leq K\right) = 1 \tag{4.3.23}$$

for some $K < \infty$.

(**i**) First show that it suffices to prove that there exists a $K < \infty$ such that

$$\mathbb{P}\left(\varlimsup_{\delta \searrow 0} \sup_{\substack{0<t-s\leq\delta \\ s,t\in[0,1]}} \frac{|B(t)-B(s)|}{L(\delta)} \leq K\right) = 1$$

and that this will follow from

$$\sum_{n=0}^{\infty} \mathbb{P}\left(\sup_{2^{-n-1}\leq t-s\leq 2^{-n}} \frac{|B(t)-B(s)|}{L(2^{-n-1})} > K\right) < \infty. \tag{*}$$

(**ii**) Define the polygonal approximation $\{B_n(t) : t \geq 0\}$ as in § 4.3.1, set $M_n = \max_{1\leq m\leq 2^n} |B(m2^{-n}) - B((m-1)2^{-n})|$, and show that

$$\frac{|B(t)-B(s)|}{L(2^{-n-1})} \leq \frac{2\|B-B_n\|_{[0,1]}}{L(2^{-n-1})} + \frac{M_n}{L(2^{-n})} \quad \text{for } 2^{-n-1} \leq t-s \leq 2^{-n}.$$

(**iii**) Set $C = \sum_{n=0}^{\infty} \sqrt{(n+1)2^{-n}}$, show that $\sum_{n=m}^{\infty} L(2^{-n})^{-1} \le CL(2^{-m})^{-1}$ for all $m \ge 0$, and, arguing as in the proof of Theorem 4.3.2, conclude that, for any $R > 0$,

$$\mathbb{P}\big(\|B - B_n\|_{[0,1]} \ge R\big) \le \sum_{m=n}^{\infty} \mathbb{P}\big(M_{m+1} \ge C^{-1} R L(2^{-m-1})^{-1}\big).$$

(**iv**) Show that, for all $R > 0$,

$$\mathbb{P}\big(M_n \ge R L(2^{-n})^{-1}\big) \le 2^{n(1-2^{-1}R^2)},$$

and combine this with (**ii**) and (**iii**) to prove that (*) holds for some $K < \infty$.