Homework #2:

Exercise 2.1:

(i) Suppose that \( \{Y_n : n \geq 1\} \subseteq \mathfrak{S} \) and that \( Y_n \rightarrow Y \) in \( L^2(\mathbb{P};\mathbb{R}) \). Then \( m_n \equiv \mathbb{E}^\mathbb{P}[Y_n] \rightarrow m \equiv \mathbb{E}^\mathbb{P}[Y] \) and \( V_n \equiv \text{Var}(Y_n) \rightarrow V \equiv \text{Var}(Y) \), and therefore

\[
\mathbb{E}^\mathbb{P}[e^{i\xi Y}] = \lim_{n \to \infty} \mathbb{E}^\mathbb{P}[e^{i\xi Y_n}] = \lim_{n \to \infty} e^{i\xi m_n + \frac{i}{2} V_n} = e^{i\xi m + \frac{i}{2} V}.
\]

Thus \( Y \) is Gaussian, and so \( \mathfrak{S} \) is a Gaussian family. The remaining assertions are trivial.

(ii) Clearly \( \sigma(L) = \sigma(\tilde{L}) \). Next,

\[
X - \Pi_{1 \otimes L}X = \tilde{X} + \mathbb{E}^\mathbb{P}[X] - \Pi_{1 \otimes L} \tilde{X} - \mathbb{E}^\mathbb{P}[X] = \tilde{X} - \Pi_{1 \otimes L} \tilde{X}.
\]

Since \( \tilde{L} \subseteq \mathbf{1} \otimes L \), we will know that \( \Pi_{1 \otimes L} \tilde{X} = \Pi_L \tilde{X} \) once we show that \( \tilde{X} - \Pi_L \tilde{X} \perp \mathbf{1} \otimes L \). But \( \mathbf{1} \perp \tilde{L} \). Thus if \( Y \in \mathbf{1} \otimes L \) and \( \tilde{Y} = Y - \mathbb{E}^\mathbb{P}[Y] \), then \( \tilde{Y} \in \tilde{L} \) and so

\[
\mathbb{E}^\mathbb{P}[|\tilde{X} - \Pi_L \tilde{X} Y|] = \mathbb{E}^\mathbb{P}[|\tilde{X} - \Pi_L \tilde{X} \tilde{Y}|] + \mathbb{E}^\mathbb{P}[Y] \mathbb{E}^\mathbb{P}[|\tilde{X} - \Pi_L \tilde{X}|] = 0.
\]

Finally, by Lemma 2.1.1, \( \tilde{X} - \Pi_L \tilde{X} \) is independent of \( \sigma(\tilde{L}) = \sigma(L) \).

Exercise 2.2: If \( 0 < s < t \), then \( B(t) - B(s) \) is independent of \( \mathcal{F}_s \) and therefore

\[
\mathbb{E}^\mathbb{P}[E_\xi(t) | \mathcal{F}_s] = E_\xi(s) e^{-\frac{\xi^2}{2} (t-s)} \mathbb{E}^\mathbb{P}[e^{\xi(B(t)-B(s))} | \mathcal{F}_s] = E_\xi(s).
\]

The reasoning given shows that \( \mathbb{P}(\|B(\cdot)\|_{[0,t]} \geq R) \leq 2 \exp(-\xi R + \frac{\xi^2 t}{2}) \) for all \( \xi \geq 0 \), and so the \( \mathbb{P}(\|B(\cdot)\|_{[0,t]} \geq R) \leq 2e^{-\frac{R^2}{2}} \) follows when one takes \( \xi = \frac{R}{t} \). When \( N \geq 2 \), observe that \( \|B(\cdot)\|_{[0,t]} \leq N_\frac{1}{2} \max_{1 \leq j \leq N} \|B(\cdot)\|_{[0,t]} \), and therefore

\[
\mathbb{P}(\|B(\cdot)\|_{[0,t]} \geq R) \leq \mathbb{P}\left( \max_{1 \leq j \leq N} \|B(\cdot)\|_{[0,t]} \geq N_\frac{1}{2} R \right) \\
\leq N \max_{1 \leq j \leq N} \mathbb{P}(\|B(\cdot)\|_{[0,t]} \geq N_\frac{1}{2} R) \leq 2Ne^{-\frac{R^2}{2}}.
\]

Exercise 2.4:

(i) If \( (X(t), \mathcal{F}_t, \mathbb{P}) \) is a Brownian motion, then the argument given above in Exercise 2.2 shows that \( (e^{i\mathbf{\xi}X(t))_{kN} + \frac{\mathbf{\xi}^2}{2}, \mathcal{F}_t, \mathbb{P}) \) is a martingale. Conversely, if it is a martingale, then

\[
\mathbb{E}^\mathbb{P}[e^{i\mathbf{\xi}X(t-X(s))_{kN} | \mathcal{F}_s} = e^{-\frac{\xi^2}{2} (t-s)},
\]

and so \( (X(t), \mathcal{F}_t, \mathbb{P}) \) is a Brownian motion.

(ii) By Hunt’s stopping time theorem,

\[
\mathbb{E}^\mathbb{P}[e^{i\mathbf{\xi}(B(t+\zeta))_{kN} + \frac{\mathbf{\xi}^2}{2} (t+\zeta) | \mathcal{F}_{s+\zeta}}] = e^{i\mathbf{\xi}(B(s+\zeta))_{kN} + \frac{\mathbf{\xi}^2}{2} (s+\zeta)}.
\]

Thus, since \( \zeta \) and \( B(\zeta) \) are \( \mathcal{F}_{s+\zeta} \)-measurable,

\[
\mathbb{E}^\mathbb{P}[e^{i\mathbf{\xi}(B(t+\zeta) - B(\zeta))_{kN} + \frac{\mathbf{\xi}^2}{2} t | \mathcal{F}_{s+\zeta}}] = e^{i\mathbf{\xi}(B(s+\zeta) - B(\zeta))_{kN} + \frac{\mathbf{\xi}^2}{2} t},
\]

and so, by (i), \( (B(t+\zeta) - B(\zeta), \mathcal{F}_{t+\zeta}, \mathbb{P}) \) is a Brownian motion.
(iii) Let $A \in \mathcal{F}_s$, and check that $A \cap \{\zeta > s\} \in \mathcal{F}_{s \wedge \zeta} \subseteq \mathcal{F}_s \cap \mathcal{F}_{t \wedge \zeta}$. Using Hunt's stopping time theorem, one has
\[
\mathbb{E}^P[e^{i(\xi, \bar{B}(t))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A] = \mathbb{E}^P[e^{i(\xi, 2B(s \wedge \zeta) - B(t))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A \cap \{\zeta \leq s\}]
+ \mathbb{E}^P[e^{i(\xi, 2B(t \wedge \zeta) - B(t))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A \cap \{\zeta > s\}]
= \mathbb{E}^P[e^{i(\xi, 2B(s \wedge \zeta) - B(s))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A \cap \{\zeta \leq s\}]
+ \mathbb{E}^P[e^{i(\xi, B(t \wedge \zeta))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A \cap \{\zeta > s\}]
= \mathbb{E}^P[e^{i(\xi, B(s))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A \cap \{\zeta \leq s\}] + \mathbb{E}^P[e^{i(\xi, B(s))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A \cap \{\zeta > s\}]
= \mathbb{E}^P[e^{i(\xi, B(s))_{kn} + \frac{|\mathbf{s}|^2}{2} t}, A].
\]
Hence, by (i), $(\bar{B}(t), \mathcal{F}_t, \mathbb{P})$ is a Brownian motion.

(iv) & (v) Just follow the steps outlined.

(vi) Since $(B(t)_{N+1}, \mathcal{F}_t, \mathbb{P})$ is an $\mathbb{R}$-valued Brownian motion, we know from (v)
that $\mathbb{P}(\zeta \leq t) = 2\mathbb{P}(B(t)_{N+1} \geq a)$. In particular, this means that $\zeta < \infty$ (a.s., $\mathbb{P}$).
Further, because $\{B(t)_{N+1} : t \geq 0\}$ is independent of $\{B(t)_j : 1 \leq j \leq N \wedge 1 \leq j \leq N\}$, the calculation in Exercise 1.4 justifies to
\[
\mathbb{P}(X \in \Gamma) = \int_0^\infty \mathbb{P}(|B(t)_1, \ldots, B(t)_N| \in \Gamma) \mathbb{P}(\zeta = dt)
\geq \frac{2a}{\sqrt{2\pi}} \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{1}{2} t \gamma_0, dt} \mathbb{P}(\zeta = dt) = \frac{2a}{\omega_N} \int_\Gamma (a^2 + |y|^2)^{-\frac{N+1}{2}} dy.
\]

Exercise 2.5: Set $B_{m,n} = B(m2^{-n})$ and $\Delta_{m,n} = B_{m+1,n} - B_{m,n}$. Clearly,
$B_{m+1,n}^2 - B_{m,n}^2 = \Delta_{m,n}^2 + 2B_{m,n}\Delta_{m,n} = -\Delta_{m,n}^2 + 2B_{m+1,n}\Delta_{m,n}$.

From these, one has
$B(1)^2 = \sum_{m=0}^{2^n-1} \Delta_{m,n}^2 + 2\sum_{m=0}^{2^{n-1}} B_{m,n}\Delta_{m,n}$ and $B(1)^2 = -\sum_{m=0}^{2^n-1} \Delta_{m,n}^2 + 2\sum_{m=0}^{2^n-1} B_{m+1,n}\Delta_{m,n}$,
and so, by (2.1.2), the first and third equations are proved. To prove the second equation, note that
$B_{m+1,n}^2 - B_{m,n}^2 - 2B_{m+1,n+1}\Delta_{m,n} = (B_{m+1,n} + B_{m,n} - 2B_{m+1,n+1})\Delta_{m,n}$
$= \Delta_{m+1,n+1}\Delta_{m,n} - \Delta_{m,n} = \Delta_{2m+1,n+1}\Delta_{m,n} = \Delta_{2m+1,n+1}^2 - \Delta_{2m,n+1}^2$.

Next, proceeding in exactly the same way as in the derivation of (2.1.2), one sees that
\[
\lim_{n \to \infty} \sum_{m=0}^{2^n-1} \Delta_{2m+1,n+1} = \frac{1}{2} = \lim_{n \to \infty} \sum_{m=0}^{2^n-1} \Delta_{2m,n+1}^2,
\]
and therefore, after another application of (2.1.2), that
$B(1)^2 = 2 \lim_{n \to \infty} \sum_{m=0}^{2^n-1} B_{2m+1,n+1}\Delta_{m,n}$. 