1 Infinitely Divisible Flows on $\mathcal{M}_1(\mathbb{R}^n)$:

In preparation for our analysis of the tangent space to $\mathcal{M}_1(\mathbb{R}^n)$ at $\delta_0$, we will need to review some classical facts about the semigroup structure of $\mathcal{M}_1(\mathbb{R}^n)$. In particular, because they play the role of rays in $\mathcal{M}_1(\mathbb{R}^n)$, we will need to recall the notion of an infinitely divisible flow on $\mathcal{M}_1(\mathbb{R}^n)$.

It is probably unnecessary to mention, but $\mathbb{R}^n$ is an Abelian group under addition, and the associated convolution product $\mathcal{M}_1 \ast \mathcal{M}_2$ of finite, Borel measures $\mathcal{M}_1$ and $\mathcal{M}_2$ is determined by

\[
\mathcal{M}_1 \ast \mathcal{M}_2(\Gamma) = \int_{\mathbb{R}^n \times \mathbb{R}^n} 1_{\Gamma}(x + y) \, M_1(dx) M_2(dy), \quad \text{for } \Gamma \in \mathcal{B}_{\mathbb{R}^n}.
\]

In particular, $\mathcal{M}_1(\mathbb{R}^n)$ becomes an Abelian semigroup under convolution. In this connection, the Fourier transform $\hat{\mu} \in \mathcal{C}_b(\mathbb{R}^n; \mathbb{C})$ of $\mu \in \mathcal{M}_1(\mathbb{R}^n)$ is given by

\[
(1) \quad \hat{\mu}(\xi) = \int_{\mathbb{R}^n} e_{\xi}(y) \, \mu(dy), \quad \text{where } e_{\xi}(y) \equiv \exp \left( \sqrt{-1} (\xi, y)_{\mathbb{R}^n} \right),
\]

and

\[
\hat{\mu} \ast \hat{\nu} = \hat{\mu} \hat{\nu} \quad \text{for all } \mu, \nu \in \mathcal{M}_1(\mathbb{R}^n).
\]

Finally, for each $\varphi \in L^1(\mathbb{R}^n; \mathbb{C}) \cap \mathcal{C}_b(\mathbb{R}^n; \mathbb{C})$ whose Fourier transform

\[
\hat{\varphi}(\xi) \equiv \int_{\mathbb{R}^n} e_{\xi}(y) \, \varphi(y) \, dy
\]

is in $L^1(\mathbb{R}^n; \mathbb{C})$: (cf. Lemma 2.2.8 in [36])

\[
(2) \quad \langle \varphi, \mu \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \hat{\mu}(-\xi) \, d\xi.
\]

A curve $t \in [0, \infty) \mapsto \lambda_t \in \mathcal{M}_1(\mathbb{R}^n)$ is said to be an infinitely divisible flow if $t \mapsto \lambda_t$ is continuous and $\lambda_{s+t} = \lambda_s \ast \lambda_t$ for all $s, t \geq 0$. Notice that, as distinguished from the situation on $\mathbb{Z}_n$, all infinitely divisible flows on $\mathbb{R}^n$ start at 0. Indeed, because $\lambda_0 = \lambda_0 \ast \lambda_0$, $\lambda_0$ must be the normalized Haar measure for some subgroup of $\mathbb{R}^n$, and $\{0\}$ is the only subgroup of $\mathbb{R}^n$ which admits a normalized Haar measure. We now want to show that each infinitely divisible flow $t \mapsto \lambda_t$ is differentiable and that the form of the tangent $\lambda_t$ is well understood. For this purpose, recall that a Lévy system is a triple $(a, b, M)$ consisting of a non-negative definite, $n \times n$ symmetric matrix $a$, a

\footnote{When $E$ is a topological space, $\mathcal{B}_E$ is the Borel field over $E$.}
vector \( b \in \mathbb{R}^n \), and a Lévy measure \( M \): a (non-negative) Borel measure \( M \) on \( \mathbb{R}^n \) with the properties that \( M(\{0\}) = 0 \) and that \( y \mapsto \frac{|y|^2}{1+|y|^2} \) is \( M \)-integrable.

We associate with a Lévy system \((a, b, M)\) the (pseudodifferential) operator \( L^{(a, b, M)} \) given by

\[
\begin{split}
L^{(a, b, M)}(x) &= \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j} \partial_i \partial_j \varphi(x) + \sum_{i=1}^{n} b_i \partial_i \varphi(x) \\
&+ \int_{\mathbb{R}^n} \left( \varphi(y + x) - \varphi(x) - \frac{\langle y, \nabla \varphi(x) \rangle_{\mathbb{R}^n}}{1 + |y|^2} \right) M(dy).
\end{split}
\]

(3)

Our main goal in this subsection is to make the preparations which will allow us to show that each infinitely divisible flow \( t \mapsto \lambda_t \) is differentiable and is associated with a unique Lévy system \((a, b, M)\) in such a way that

\[
\dot{\lambda}_t \varphi = \langle L^{(a, b, M)} \varphi, \lambda_t \rangle \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \varphi \in C^2_b(\mathbb{R}^n; \mathbb{C}).
\]

(4)

As the cognoscenti already realize, (4) is really just a restatement of the famous Lévy–Khinchine formula for infinitely divisible laws. Thus, it will come as no surprise to them that what we are about to do yields a proof of that formula.

**Lemma 5.** Given a Lévy system \((a, b, M)\), the operator \( L \) in (3) is well defined on \( C^2_b(\mathbb{R}^n; \mathbb{C}) \) and, for each \( R \in (0, \infty) \), satisfies the estimate\(^2\)

\[
|L^{(a, b, M)}(x)| \leq C \|\varphi\|_{C^2_b(B(x, R) ; \mathbb{C})} + M(B(x, R) \mathbb{C}) \|\varphi\|_u.
\]

(6)

In particular, \( \varphi \in C^2_b(\mathbb{R}^n; \mathbb{C}) \mapsto L^{(a, b, M)} \varphi \in C_b(\mathbb{R}^n; \mathbb{C}) \) is continuous. Finally, if \( \varphi \) is in the Schwartz test function class\(^3\) \( S(\mathbb{R}^n; \mathbb{C}) \), then\(^4\)

\[
L^{(a, b, M)}(\xi) = \ell^{(a, b, M)}(-\xi) \hat{\varphi}(\xi) \quad \text{where}
\]

\[
\ell^{(a, b, M)}(\xi) = -\frac{(\xi, a\xi)_{\mathbb{R}^n}}{2} + \sqrt{-1} (b, \xi)_{\mathbb{R}^n}
\]

\[
+ \int_{\mathbb{R}^n} \left( \xi(y) - 1 - \frac{\sqrt{-1}(\xi, y)_{\mathbb{R}^n}}{1 + |y|^2} \right) M(dy).
\]

(7)

\(^2\) We use \( \|\cdot\|_u \) to denote the uniform or supremum norm of a function.

\(^3\) The class of smooth functions \( \varphi \) with the property that \( x \mapsto (1 + |x|^2)^m \partial^n \varphi(x) \) is bounded for each \( m \in \mathbb{N} \) and all \( \alpha \in \mathbb{N}^n \).

\(^4\) In the jargon of pseudodifferential operator theory, the statement which follows identifies \( \xi \mapsto \ell^{(a, b, M)}(-\xi) \) as the symbol of \( L^{(a, b, M)} \).
Proof: As an application of Taylor’s theorem, one knows that there exists a \( \kappa < \infty \), depending only on \( n \), such that

\[
\left| \varphi(y + x) - \varphi(x) - \frac{\langle y, \nabla \varphi(x) \rangle_{\mathbb{R}^n}}{1 + |y|^2} \right| \leq \frac{\kappa |y|^2}{1 + |y|^2} \| \varphi \|_{C^2_b(\mathbb{R}^n; \mathbb{C})}.
\]

Thus, both the definition of \( L^{(a, b, M)} \) on \( C^2_b(\mathbb{R}^n; \mathbb{C}) \) is justified and the bound (6) is clear. Furthermore, if, for each \( \epsilon > 0 \), \( M^\epsilon(dy) \equiv 1_{(\epsilon, \infty)}(|y|)M(dy) \), then, for each \( \epsilon > 0 \), there is no question that \( L^{(a, b, M')} \varphi \) determines a continuous map from \( C^2_b(\mathbb{R}^n; \mathbb{C}) \) into \( C_b(\mathbb{R}^n; \mathbb{C}) \). Hence, because

\[
|L^{(a, b, M)} \varphi(x) - L^{(a, b, M')} \varphi(x)| \leq \kappa \| \varphi \|_{C^2_b(\mathbb{R}^n; \mathbb{C})} \int_{|y|<\epsilon} \frac{|y|^2}{1 + |y|^2} M(dy),
\]

it is clear that the same continuity result extends to \( \varphi \sim L^{(a, b, M)} \varphi \). Finally, (7) is an elementary application of the basic theory of the Fourier transform and is therefore left to the reader. \( \square \)

We next give an operator theoretic characterization of Lévy systems.

Lemma 8. Use \( D(\mathbb{R}^n; \mathbb{R}) \) to denote the space of functions \( \varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) for which there exists a \( \varphi(\infty) \in \mathbb{R} \) with the property that \( \varphi - \varphi(\infty) \) has compact support. Then a linear functional \( A : D(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R} \) is given by (cf. (3)) \( A \varphi = L^{(a, b, M)} \varphi(0) \) for some Lévy system \((a, b, M)\) if and only if \( A \) satisfies

1. The minimum principle: \( A \varphi \geq 0 \) whenever \( 0 \) is a global minimum of \( \varphi \in D(\mathbb{R}^n; \mathbb{R}) \),
2. Tightness: for any \( \varphi \in D(\mathbb{R}^n; \mathbb{R}) \), \( \lim_{R \to \infty} A \varphi_R = 0 \) where \( \varphi_R(y) \equiv \varphi(R^{-1}y) \).

Proof: First observe that, for any Lévy system \((a, b, M)\), \( \varphi \sim L^{(a, b, M)} \varphi(0) \) satisfies both the minimum principle and the tightness property. Indeed, the tightness property is an easy variant on the argument to prove (6): by Taylor’s theorem, for any \( R \geq 1 \):

\[
|L^{(a, b, M)} \varphi_R(0)| \leq \| \varphi \|_{C^2_b(\mathbb{R}^n; \mathbb{R})} \left( R^{-2} \|a\|_{op} + R^{-1} |b| \right) + 3M(B_{\mathbb{R}^n}(0, \sqrt{R} \mathbb{C})
\]

\[
+ \int_{|y| \leq \sqrt{R}} \left( R^{-2} |y|^2 + R^{-1} M(dy) \right) M(dy).
\]

As for the minimum principle, note that if \( 0 \) is a global minimum of \( \varphi \in D(\mathbb{R}^n; \mathbb{R}) \), then, by the first and second order derivative tests, its gradient
vanishes at 0 and its Hessian is non-negative definite there. Hence, because 
\(a\) is non-negative definite, 
\[
L^{(a,b,M)}(0) \geq \int_{\mathbb{R}^n} \left( \varphi(y) - \varphi(0) \right) M(dy) \geq 0.
\]

Now suppose that \(A\) is a linear functional on \(D(\mathbb{R}^n; \mathbb{R})\) which satisfies the minimum principle and the tightness condition. Let \(\psi \in C^\infty(\mathbb{R}; [0,1])\) be a function with the properties that \(\psi \mid (-\infty, 0] = 1\), \(\psi \mid [1, 2] = 0\) is nonincreasing, and \(\psi \mid [2, \infty) = 0\). For \(R > 0\), define \(\eta_R \in C^\infty_c(\mathbb{R}^n; [0,1])\) by \(\eta_R(x) = \psi(R^{-1}|x|)\) and \(\varphi \leadsto A^R \varphi\) so that \(A^R \varphi = A((1 - \eta_R)\varphi)\). By the minimum principle, we know that \(\varphi \geq 0 \implies A^R \varphi \geq 0\), and so \(\varphi_1 \leq \varphi_2 \implies A^R \varphi_1 \leq A^R \varphi_2\). In particular, for \(r \geq 2R\),

\[
|A^R \varphi| \leq \|\varphi\|_{C_b(B_{2R}(0,2r); \mathbb{R})} A((1 - \eta_R)\eta)_R \|\varphi\|_{C_b(B_{R}(0,2r); \mathbb{R})} A(1 - \eta_R),
\]

and therefore, by the tightness property, for each \(\epsilon > 0\) we can choose \(r_\epsilon \in (2R, \infty)\) so that

\[
|A^R \varphi| \leq K_R \|\varphi\|_{C_b(B_{R}(0,r_\epsilon); \mathbb{C})} + \epsilon \|\varphi\|_{C_b(B_{R}(0,r_\epsilon); \mathbb{C})},
\]

where \(K_R = A(1 - \eta_R)\). This means, by the form of the Riesz representation theorem in Lemma 3.1.7 of [36], there exists a finite Borel measure \(M^R\) on \(\mathbb{R}^n\) such that \(A^R \varphi = \int_{\mathbb{R}^n} \varphi(y) M^R(dy)\). Moreover, because \(A^{R_1} \varphi = A^{R_2} \varphi\) if \(R_1 \leq R_2\) and \(\varphi\) vanishes on \(B_{R_n}(0,2R_2)\), we have now proved that there is a unique Borel measure \(M\) on \(\mathbb{R}^n\) such that \(M(\{0\}) = 0\) and

\[
(*) \quad A((1 - \eta_R)\varphi) = \int_{\mathbb{R}^n} (1 - \eta_R(y)) \varphi(y) M(dy) \quad \text{for all } R > 0.
\]

Next, given \(\xi \in \mathbb{R}^n\), set \(\varphi_\xi(y) = (\xi, y)_{\mathbb{R}^n}\), and apply the minimum principle to see that

\[
|\xi|^2 A(|\xi|^2 \eta_1) \geq A(\varphi_\xi^2 \eta_{R_1}) \geq A(\varphi_\xi^2 \eta_{R_2}) \geq 0 \quad \text{for } 0 < R_1 \leq R_2 \leq 1.
\]

Hence, by polarization, we conclude that there exists a symmetric, non-negative definite \(a \in \text{Hom}(\mathbb{R}^n; \mathbb{R}^n)\) such that

\[
(**) \quad \lim_{R \searrow 0} A(\varphi_\xi \varphi_{\xi'} \eta_R) = (\xi, a \xi')_{\mathbb{R}^n} \quad \text{for all } \xi, \xi' \in \mathbb{R}^n.
\]

In addition, if \(\varphi\) and all its derivatives of first and second order vanish at 0, then there exists a \(C < \infty\) for which \(|\varphi(y)| \eta_R(y) \leq CR |y|^2 \eta_R(y)\) and so

\[
(***) \quad \lim_{R \searrow 0} A(\varphi \eta_R) = 0 \quad \text{when } \partial^\alpha \varphi(0) = 0 \text{ for all } |\alpha| \leq 2.
\]
To complete the proof from here, let \( \varphi \in D(\mathbb{R}^n; \mathbb{C}) \) be given, and, for \( R > 0 \), write \( \varphi(y) \) as

\[
\begin{align*}
\varphi(0) + \eta_1(y)(y, \text{grad}\varphi(0))_{\mathbb{R}^n} + \frac{1}{2} \eta_R(y)(y, \text{Hess}\varphi(0)y)_{\mathbb{R}^n} \\
+ \eta_R(\varphi(y) - \varphi(0) - \eta_1(y)(y, \text{grad}\varphi(0))_{\mathbb{R}^n} - \frac{1}{2}(y, \text{Hess}\varphi(0)y)_{\mathbb{R}^n}) \\
+ (1 - \eta_R(y))(\varphi(y) - \varphi(0) - \eta_1(y)(y, \text{grad}\varphi(0))_{\mathbb{R}^n})
\end{align*}
\]

where \( \text{Hess}\varphi \) denotes the Hessian of \( \varphi \). By the minimum principle applied to \( 1 \) and \(-1\), we know that \( A \) annihilates constants. Thus, by linearity,

\[
A\varphi = \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j} \partial_{x_i} \partial_{x_j} \varphi(0) + \sum_{i=1}^{n} \tilde{b}_i \partial_{x_i} \varphi(0) + \eta_R(\tilde{\varphi}) \\
+ \int_{\mathbb{R}^n} \left( 1 - \eta_R(y) \right) \left( \varphi(y) - \varphi(0) - \eta_1(y)(y, \text{grad}\varphi(0))_{\mathbb{R}^n} - \frac{1}{2}(y, \text{Hess}\varphi(0)y)_{\mathbb{R}^n} \right) M(dy),
\]

where \( \tilde{b}_i = A(\eta_1 \varphi_i) \) and \( a_{i,j} = A(\eta_R \varphi_i \varphi_j) \) with \( \varphi_i(y) = y_i \)

when \( y = (y_1, \ldots, y_n) \) and

\[\tilde{\varphi}(y) = \varphi(y) - \varphi(0) - \eta_1(y)(y, \text{grad}\varphi(0))_{\mathbb{R}^n} - \frac{1}{2}(y, \text{Hess}\varphi(0)y)_{\mathbb{R}^n} \cdot\]

Hence, by (*), (**), and (***) , we obtain

\[
A\varphi = \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j} \partial_{x_i} \partial_{x_j} \varphi(0) + \sum_{i=1}^{n} \tilde{b}_i \partial_{x_i} \varphi(0) \\
+ \int_{\mathbb{R}^n} \left( \varphi(y) - \varphi(0) - \eta_1(y)(y, \text{grad}\varphi(0))_{\mathbb{R}^n} - \frac{1}{2}(y, \text{Hess}\varphi(0)y)_{\mathbb{R}^n} \right) M(dy)
\]

after letting \( R \downarrow 0 \). Starting from here, it is easy to check that \( \frac{|y|^2}{1+|y|^2} \) is \( M \)-integrable and that \( A\varphi = L(\alpha, \beta, \sigma, \mu) \varphi(0) \) where

\[
b = \tilde{b} + \int_{\mathbb{R}^n} \left( \frac{y}{1+|y|^2} - \eta_1(y)y \right) M(dy).
\]

With these preparations, we can now prove the Lévy–Khinchine formula.\(^5\)

\(^5\) There are many proofs of this renowned formula, most of which rely more heavily on Fourier analysis. For example, see §3.2 in [36] for the case when \( n = 1 \) and Chapter VI of [26] for the result in the setting of a general locally compact, Abelian group.
Theorem 9. Suppose that \( \lambda \in M_1(\mathbb{R}^n) \) is infinitely divisible in the sense that, for each \( m \geq 1 \), there is an \( \lambda_{1/m} \in M_1(\mathbb{R}^n) \) such that \( \lambda = \lambda_{1/m} \). Then there exists a unique Lévy system \( (a, b, M) \) with the property that (cf. (7))

\[
\hat{\lambda}(\xi) = f(a, b, M) \equiv e^{\ell(a, b, M)(\xi)}.
\]

Conversely, if \( (a, b, M) \) is a Lévy system, then there is a unique probability measure \( \lambda(a, b, M) \) which for which (10) holds with \( \lambda = \lambda(a, b, M) \), and \( \lambda(a, b, M) \) is infinitely divisible.

Proof: Because it is the easier part, we begin with the converse statement. First note that it suffices to prove that, for each \( (a, b, M) \), \( f(a, b, M)(\xi) \) is the Fourier transform of a probability measure, since infinite divisibility will then follow when one replaces \( (a, b, M) \) by \( (\frac{1}{m}a, \frac{1}{m}b, \frac{1}{m}M) \). Next, observe that it suffices to treat the cases \( (a, b, 0) \) and \( (0, 0, M) \) separately, since, if \( \lambda(a, b, 0) \) and \( \lambda(0, 0, M) \) exist, then \( \lambda(a, b, M) \) exists and is equal to \( \lambda(a, b, 0) \ast \lambda(0, 0, M) \). In addition, \( f(a, b, 0) \) is easily recognized as the Fourier transform of the Gauss measure on \( \mathbb{R}^n \) with mean \( b \) and covariance \( a \). Thus, everything comes down to proving the existence of \( \lambda(0, 0, M) \). To this end, define \( \lambda^* \) as in the proof of Lemma 5, set \( \beta^* = -\int_{\mathbb{R}^n} \frac{y}{|y|} \mathcal{M}(dy) \), and define

\[
\lambda^* = \delta_{\beta^*} \ast \left( e^{-M^*(\mathbb{R}^n)} \sum_{m=0}^{\infty} \frac{1}{m!} (M^*)^{*m} \right).
\]

It is then an easy matter to check that \( \lambda^* \in M_1(\mathbb{R}^n) \) and that \( \hat{\lambda}^*(\xi) = f(0, 0, M^*)(\xi) \). Finally, observe that \( \ell(0, 0, M^*) \longrightarrow \ell(0, 0, M) \) uniformly on compacts, and apply Lévy’s continuity theorem (cf. Exercise 3.1.19 in [36]) to conclude that \( \lambda^* \longrightarrow \lambda^0 \) in \( M_1(\mathbb{R}^n) \), where \( \lambda^0 = f(0, 0, M) \). That is, \( \lambda^0 = \lambda(0, 0, M) \).

We now return to first part of the theorem. Thus, suppose that \( \lambda \) is infinitely divisible, set \( f = \hat{\lambda} \), and let \( \lambda_{1/m} \) be given accordingly. Our strategy for showing that \( \lambda = \lambda(a, b, M) \) for some Lévy system \( (a, b, M) \) will be to prove that

\[
A\varphi \equiv \lim_{m \to \infty} m(\langle \varphi, \lambda_{1/m} \rangle - \varphi(0))
\]

exists for \( \varphi \) from the class \( D(\mathbb{R}^n; \mathbb{R}) \) considered in Lemma 8 and to verify that the resulting linear functional \( A \) satisfies the minimum principle and the tightness condition described in that lemma.

The first step is to show that \( \lambda_{1/m} \) tends to \( \delta_0 \) in \( M_1(\mathbb{R}^n) \) as \( m \to \infty \). For this purpose, choose \( \delta > 0 \) so that \( |\xi| \leq \delta \implies |f(\xi) - 1| \leq \frac{1}{2} \). It is then clear that

\[
\hat{\lambda}_{1/m}(\xi) = g_{1/m}(f(\xi) - 1), \quad |\xi| \leq \delta,
\]
where, for any $\alpha \in \mathbb{R}$,

$$g_{\alpha}(z) \equiv 1 + \sum_{j=1}^{\infty} \left( \frac{\alpha}{j} \right) z^j \quad \text{with} \quad \left( \frac{\alpha}{j} \right) \equiv \frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j!}$$

is the branch of the $z \mapsto (1+z)^{\alpha}$ on the unit disk which is 1 at the origin. In particular, this means first that

$$|\xi| \leq \delta \implies |1 - \widehat{\lambda}_{1/m}(\xi)| \leq \frac{1}{m}$$

and then that, for any $e \in \mathbb{S}^{n-1}$ and $r > 0$,

$$\frac{1}{m} \geq \frac{1}{\delta} \int_{0}^{\delta} \left( 1 - \Re(\widehat{\lambda}_{1/m}(te)) \right) dt = \int_{\mathbb{R}^n} \left( 1 - \frac{\sin(\delta(e,y)_{\mathbb{R}^n})}{\delta(e,y)_{\mathbb{R}^n}} \right) \lambda_{1/m}(dy) \geq \epsilon(\delta r) \lambda_{1/m}(\{y : |(y,e)_{\mathbb{R}^n}| \geq r\}),$$

where $\epsilon(\beta) \equiv \inf \left\{1 - \frac{\sin t}{t} : |t| \geq \beta \right\} > 0$ for $\beta > 0$.

As a consequence, we see that $\lambda_{1/m} \to \delta_0$.

In view of the preceding, we know that, as $m \to \infty$, $\widehat{\lambda}_{1/m}$ tends to 1 uniformly on compacts as $m \to \infty$. Hence, since $f = (\lambda_{1/m})^m$ for all $m \geq 1$, $f$ cannot vanish anywhere, and therefore there exists a unique continuous $\ell : \mathbb{R}^n \to \mathbb{C}$ such that $\ell(0) = 0$ and $f(\xi) = e^{\ell(\xi)}$. Similarly, $\widehat{\lambda}_{1/m}$ never vanishes for any $m$, and so there exists a unique continuous $\ell_{1/m}$ such that $\ell_{1/m}(0) = 0$ and $\widehat{\lambda}_{1/m} = e^{\ell_{1/m}}$. But this means that and $\xi \mapsto (\sqrt{-1}2\pi)^{-1}(\ell(\xi) - m\ell_{1/m}(\xi))$ is a continuous $\mathbb{Z}$-valued function, and so, since $\ell(0) = 0 = m\ell_{1/m}(0)$, we now know that $\widehat{\lambda}_{1/m}(\xi) = e^{\ell_{1/m}}$ for all $m \geq 1$ and $\xi \in \mathbb{C}$.

The next step is to show that

$$f_m(\xi) \equiv \exp \left( \int_{\mathbb{R}^n} (e_{\xi}(y) - 1) \lambda_{1/m}(dy) \right) \to f(\xi)$$

uniformly on compacts. For this purpose, first note that, for any $R > 0$ and $m \geq R^2$,}

$$|f_m(\xi) - f(\xi)| = \left| e^{-m} \sum_{k=0}^{\infty} \frac{m^k}{k!} \left( \widehat{\lambda}_{1/m}(\xi)^k - \widehat{\lambda}_{1/m}(\xi)^m \right) \right| \leq 2e^{-m} \sum_{\{k \geq 0 : |k-m| \geq R\sqrt{m}\}} \frac{m^k}{k!} + \sup_{0 \leq k \leq R\sqrt{m}} \left| \widehat{\lambda}_{1/m}(\xi)^k - 1 \right|.$$
By Chebychev’s inequality for Poisson random variables, one knows that
\[ e^{-m} \sum_{\{k \geq 0: |k - m| \geq R\sqrt{m}\}} \frac{m^k}{k!} \leq R^{-2} \quad \text{for all } m \geq 1. \]

Finally, by the conclusion reached at the end of the preceding paragraph, we know that, for each \( R > 0 \), as \( m \to \infty \), \( \sup_{k \leq R\sqrt{m}} |\hat{\lambda}_{1/m}(\xi)^k - 1| \to 0 \), uniformly on compacts. In particular, we have now shown that \( f_m \to f \) and therefore
\[ \lim_{m \to \infty} m \int_{\mathbb{R}^n} (\xi(y) - 1) \lambda_{1/m}(dy) = \ell(\xi) \]
uniformly on compacts.

The preceding allows us to prove that \( |\ell(\xi)| \leq C(1 + |\xi|^2) \) for some \( C < \infty \). Indeed, by the preceding, we know that
\[ (**): \sup_{m \geq 1} \left( \int_{|\xi| \leq 1} (1 - \cos(\xi, y)_{\mathbb{R}^n}) \lambda_{1/m}(dy) \lor \left| \int_{|\xi| \leq 1} \sin(\xi, y)_{\mathbb{R}^n} \lambda_{1/m}(dy) \right| \right) < \infty. \]

Hence, after integrating the first of these along the ray from the origin to \( \xi \in S^{n-1} \), we see that
\[ \sup_{m \geq 1} \int_{|\xi| \leq 1} \left( 1 - \frac{\sin(\xi, y)_{\mathbb{R}^n}}{(\xi, y)_{\mathbb{R}^n}} \right) \lambda_{1/m}(dy) < \infty \quad \text{for all } \xi \in S^{n-1}, \]
from which it is clear that
\[ C_1 \equiv \sup_{m \geq 1} m \int_{|y| \leq 1} |y|^2 \lambda_{1/m}(dy) < \infty \quad \text{and } C_2 \equiv \sup_{m \geq 1} m \lambda_{1/m}(B_{\mathbb{R}^n}(0, 1) \setminus C) < \infty. \]

In particular, \( |\ell(\xi)| \leq 2C_2 + \sup_{m \geq 1} |h_m(\xi)| \), where
\[ h_m(\xi) \equiv m \int_{|y| < 1} (\xi(y) - 1) \lambda_{1/m}(dy). \]

In addition, by the second estimate in (***) and the preceding, we know that
\[ C_3 \equiv \sup_{m \geq 1} \sup_{|\xi| \leq 1} m \int_{|y| < 1} \sin(\xi, y)_{\mathbb{R}^n} \lambda_{1/m}(dy) < \infty. \]

6 We take \( \frac{\sin 0}{0} = 1. \)
Hence, since $\partial_\xi h_m(0) = \sqrt{-1}m \int_{|\eta|<1}(\xi, y)_{\mathbb{R}^n} \lambda_{1/m}(dy)$,

$$|\partial_\xi h_m(0)| \leq C_3 + \left|m \int_{|\eta|<1} \left((\xi, y)_{\mathbb{R}^n} - \sin(\xi, y)_{\mathbb{R}^n}\right) \lambda_{1/m}(dy)\right| \leq C_3 + C_1$$

for all $\xi \in S^{n-1}$. At the same time, $h_m(0) = 0$ and all second derivatives of $h_m$ are uniformly bounded in absolute value by $C_1$. Thus, we have now shown that $|\ell(\xi)| \leq 2C_2 + (C_1 + C_3)|\xi| + \frac{1}{2}C_1|\xi|^2$.

We can at last show that the limit in (※) exists. In fact, given $\phi \in D(\mathbb{R}^n; \mathbb{R})$, set $\tilde{\phi} = \phi - \phi(\infty)$, and note that

$$m\left(\langle \phi, \lambda_{1/m}\rangle - \phi(0)\right) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} m(e^{\ell(\xi)} - 1)\hat{\tilde{\phi}}(\xi) d\xi,$$

and so (remember that $\Re\ell \leq 0$), by Lebesgue's dominated convergence theorem and the preceding estimate on $|\ell(\xi)|$, we see that the limit in (※) not only exists but also that

(***)

$$A\phi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \ell(-\xi)\hat{\tilde{\phi}}(\xi) d\xi,$$

In particular, because $\hat{\phi}_R(\xi) = \hat{\phi}(R\xi)$ and $\hat{\tilde{\phi}}$ is rapidly decreasing, another application of the estimate for $|\ell(\xi)|$ shows that $A$ satisfies the tightness property in Lemma 8. At the same time, from its definition in (※), it is clear that $A$ satisfies the minimum principle in that lemma. Hence, by Lemma 8, we now know that there is a unique Lévy system $(a, b, M)$ such that $A\phi = L^{(a,b,M)}\phi(0)$ for all $\phi \in D(\mathbb{R}^n; \mathbb{R})$. But, by (***), and elementary Fourier analysis, this means that, for any $\phi \in C^\infty_c(\mathbb{R}^n; \mathbb{R})$,

$$\int_{\mathbb{R}^n} \hat{\phi}(\xi)\ell^{(a,b,M)}(-\xi) d\xi = (2\pi)^n L^{(a,b,M)}\hat{\phi}(0)$$

$$= (2\pi)^n A\phi = \int_{\mathbb{R}^n} \hat{\phi}(\xi)\ell(-\xi) d\xi,$$

and from this the conclusion $\ell = \ell^{(a,b,M)}$ is an easy step. Namely, using the fact that both $|\ell(\xi)|$ and $|\ell^{(a,b,M)}(\xi)|$ have at most quadratic growth, one first extends the equality $\int_{\mathbb{R}^n} \hat{\phi}(\xi)\ell^{(a,b,M)}(-\xi) d\xi = \int_{\mathbb{R}^n} \hat{\phi}(\xi)\ell(-\xi) d\xi$ to all $\phi \in \mathcal{S}(\mathbb{R}^n; \mathbb{C})$. One then uses the fact that $\mathcal{S}(\mathbb{R}^n; \mathbb{C})$ is invariant under the Fourier transform to see that this equality holds when $\hat{\phi}$ is replaced by $\phi$, at which point the asserted conclusion becomes obvious. □