Exercise 5 for Feb 2nd

The goal of this exercise is to develop and apply a formula discovered by Euler. In what follows, for any \( t \in \mathbb{R} \), \([t] = \max \{ n \in \mathbb{Z} : n \leq t \}\) is the integer part of \( t \) and \([t] = t - t\) is the fractional part of \( t \).

Assume that \( f : [0, \infty ) \rightarrow \mathbb{C} \) is a continuously differentiable function. Show that for integers \( m < n \)

\[
\sum_{k=m}^{n} f(k) = f(m) + \int_{m}^{n} f(t) \, dt = f(m) + \int_{m}^{n} f(t) \, dt - \int_{m}^{n} f(t) \, d[t],
\]

and use this and integration by parts to conclude that

\[
(1) \quad \sum_{k=m}^{n} f(k) = f(m) + \int_{m}^{n} f(t) \, dt + \int_{m}^{n} f'(t)[t] \, dt,
\]

which is Euler's formula.

Next set \( \rho(t) = [t] - \frac{1}{2} \) and \( R(t) = \int_{0}^{t} \rho(s) \, ds \), and show that \( R(t+1) = R(t) \) and \( |R(t)| \leq \frac{1}{4} \) for all \( t \geq 0 \) and that \( R(n) = 0 \) for all \( n \in \mathbb{N} \). Assuming that \( f \) is twice continuously differentiable, use (1) and integration by parts to show that

\[
(2) \quad \sum_{k=m}^{n} f(k) = \int_{m}^{n} f(t) \, dt + \frac{f(n) + f(m)}{2} + \int_{m}^{n} f''(t)R(t) \, dt.
\]

Apply (2) to show that

\[
\sum_{k=1}^{n} \frac{1}{k} = \log n + \frac{n+1}{2n} + 2 \int_{1}^{n} \frac{R(t)}{t^3} \, dt,
\]

and use this to conclude that

\[
\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + \frac{1}{2n} + E(n),
\]

where

\[
\gamma \equiv \frac{1}{2} + 2 \int_{1}^{\infty} \frac{R(t)}{t^3} \, dt
\]

is called Euler's constant and \( |E(n)| \leq \frac{1}{8n^2} \).

Now use (2) to show that

\[
\sum_{k=1}^{n} \log k = n \log n - n + \frac{\log n}{2} - \int_{1}^{n} \frac{R(t)}{t^2} \, dt.
\]

Finally, show that

\[
- \int_{1}^{n+1} \frac{R(t)}{t^2} \, dt > - \int_{1}^{n} \frac{R(t)}{t^2} \, dt > 0,
\]

for all \( n \geq 2 \), and conclude that

\[
(3) \quad 1 \leq \frac{n!}{\sqrt{Cn}(\frac{2}{e})^n} \leq e^{\frac{1}{2n}},
\]
where

\[ C = \exp \left( -2 \int_1^\infty \frac{R(t)}{t^2} \, dt \right). \]

The result in (3) is a somewhat more precise form of a result proved by de Moivre, a contemporary of Euler. Shortly afterwards, Stirling showed that \( C = 2\pi \), and, in spite of his acknowledgment of de Moivre’s decisive contribution, Stirling ended up with the full credit. Stirling obtained his result as an application of Wallis’s formula. A more transparent derivation can be obtained as an application of the Central Limit Theorem, which is a generalization of the result for which de Moivre derived his result.