Exercise 5 for Feb 2nd

The goal of this exercise is to develop and apply a formula discovered by Euler. In what follows, for any \( t \in \mathbb{R} \), \( \lfloor t \rfloor = \max\{n \in \mathbb{Z} : n \leq t\} \) is the integer part of \( t \) and \( \lfloor t \rfloor = t - \lfloor t \rfloor \) is the fractional part of \( t \).

Assume that \( f : [0, \infty) \rightarrow \mathbb{C} \) is a continuously differentiable function. Show that for integers \( m < n \)

\[
\sum_{k=m}^{n} f(k) = f(m) + \int_{m}^{n} f(t) dt - \int_{m}^{n} f(t) d\lfloor t \rfloor,
\]

and use this and integration by parts to conclude that

\[
(1) \quad \sum_{k=m}^{n} f(k) = f(m) + \int_{m}^{n} f(t) dt + \int_{m}^{n} f'(t)\lfloor t \rfloor dt,
\]

which is Euler’s formula.

Next set \( \rho(t) = \lfloor t \rfloor - \frac{1}{2} \) and \( R(t) = -\int_{t}^{\infty} \rho(s) ds \), and show that \( R(t+1) = R(t) \) and \( 0 \leq R(t) \leq \frac{1}{8} \) for all \( t \geq 0 \). In particular, \( R(n) = 0 \) for all \( n \in \mathbb{N} \). Assuming that \( f \) is twice continuously differentiable, use (1) and integration by parts to show that

\[
(2) \quad \sum_{k=m}^{n} f(k) = \int_{m}^{n} f(t) dt + \frac{f(n) + f(m)}{2} + \int_{m}^{n} f''(t)R(t) dt.
\]

Apply (2) to show that

\[
\sum_{k=1}^{n} \frac{1}{k} = \log n + \frac{n+1}{2n} + 2 \int_{1}^{n} \frac{R(t)}{t^3} dt,
\]

and use this to conclude that

\[
\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + \frac{1}{2n} - E(n),
\]

where

\[
\gamma \equiv \frac{1}{2} + 2 \int_{1}^{\infty} \frac{R(t)}{t^3} dt
\]

is called Euler’s constant and \( 0 \leq E(n) \leq \frac{1}{8n^2} \).

Now use (2) to show that

\[
\sum_{k=1}^{n} \log k = n \log n - n + \frac{\log n}{2} - \int_{1}^{n} \frac{R(t)}{t^2} dt.
\]

Finally, show that

\[
0 \leq \int_{n}^{\infty} \frac{R(t)}{t^2} dt \leq \frac{1}{8n},
\]

for all \( n \geq 1 \), and conclude that

\[
e^{-\frac{1}{2n}} \leq \frac{n!}{\sqrt{2\pi n (\frac{n}{e})^n}} \leq 1.
\]
where

\[ C = \exp \left( 2 \int_{1}^{\infty} \frac{1 - R(t)}{t^2} \, dt \right). \]

The result in (3) is a somewhat more precise form of a result originally proved by de Moivre, a contemporary of Euler. Shortly afterwards, Stirling showed that \( C = 2\pi \), and, in spite of his acknowledgment of de Moivre's seminal contribution, Stirling ended up getting the full credit. Stirling obtained his result as an application of Wallis's formula. A more transparent derivation can be obtained as an application of the Central Limit Theorem, which is a generalization of the result for which de Moivre derived his result.