RESEARCH STATEMENT

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1. Introduction

The central object of study in my research is the *global Cherednik algebra* $\mathscr{H}_{c,\omega}$ associated to a smooth complex algebraic variety with the action of a finite group. This is a sheaf of algebras on the variety in the equivariant topology, introduced by Etingof in [2], and depending on parameters c, ω associated to the space and the group action. If the variety in question is a complex vector space affording a linear action, then the global section algebra of the sheaf $\mathscr{H}$ is the more familiar *rational Cherednik algebra* H_c, which is a rational degeneration of Cherednik's double affine Hecke algebra. A good overview of the theory of rational Cherednik algebras can be found in [4].

The global Cherednik algebra can be thought of locally, in a complex analytic neighborhood of a point, as a matrix algebra taking values in a rational Cherednik algebra associated to the tangent space at the point with action of the isotropy group.

The algebra $\mathscr{H}$ may also be though of as a deformation of the skew-product of the sheaf of differential operators on the variety with the group algebra. As such, it is natural to expect a strong link between the study of the algebra $\mathscr{H}$ and its representation theory to the theory of equivariant $\mathscr{D}$-modules.

2. Thesis

2.1. **Hecke algebras.** Cherednik algebras are also closely related to Hecke algebras via the KZ (Knizhnik-Zamolodchikov) functor. The Hecke algebra associated to a finite group is a deformation of its group algebra over $\mathbb{C}$. An important conjecture, the so-called *BMR freeness conjecture* of [1], states that this deformation is free over the parameter space, and was recently proved, see [3].

An important step in the proof, due to Losev [9], involves proving that the the KZ functor from the GGOR category $\mathscr{O}$ of [7] to the category of finite-dimensional representations of the Hecke algebra is essentially surjective.

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There is an analogous notion of Hecke algebra in the global setting as well, known as the *orbifold Hecke algebra* because it is a deformation of the orbifold fundamental group of the variety. The KZ functor in this setting essentially takes an $\mathscr{O}$-coherent $\mathscr{H}$-module to its restriction to the variety's regular locus, where it can be thought of as an equivariant flat connection the monodromy of which factors through the Hecke algebra. We prove in [11] that KZ functor from the category of $\mathscr{O}$-coherent modules for the Cherednik algebra to finite dimensional modules over the Hecke algebra is essentially surjective. Using this, we can give criteria under which the category of $\mathscr{O}$-coherent $\mathscr{H}$-module is nonzero in the case of the Riemann sphere or products of elliptic curves.

2.2. **Holonomic modules.** We have noted that we may think of the representation theory of $\mathscr{H}$ as a deformation of the theory of W-equivariant $\mathscr{D}$-modules on a W-variety. Losev [8] has introduced the notion of *holonomic* representations of certain algebras, including rational Cherednik algebras.

In [10] we generalize several basic results from the theory of (equivariant, twisted) $\mathscr{D}$-modules on varieties to the case of $\mathscr{H}$-modules. In particular, we introduce pullback and pushforward for certain equivariant maps between varieties. We also introduce the Verdier dual functor on the derived category, and discuss an analog of Kashiwara's theorem. Finally, we prove, in the case of generic parameters for the rational Cherednik algebra, that pushforward with respect to open affine inclusions takes holonomic modules to holonomic modules over the rational Cherednik algebra.

In ongoing work with Bellamy and Etingof we attempt to generalize the preservation of holonomicity under pushforward and pullback to arbitrary parameters and a more general class of morphisms. In doing so, we develop a theory of b-functions which generalizes the one for $\mathscr{D}$-modules. We will study the Goresky-MacPherson functor of minimal extension and classify irreducible holonomic modules.

In general, there is a filtration on the category of $\mathscr{H}$-modules by dimension of support. We will study this filtration, and provide a description of its associated graded in terms of $\mathscr{D}$-modules along the strata and objects of GGOR categories $\mathscr{O}$ in the normal direction. We should have versions of such a theorem for all, coherent, holonomic, and regular (defined below) holonomic modules. We develop a working "five-functor formalism" (we don't have a tensor product) for Cherednik modules on general varieties. We also prove that the dimension of Ext between two holonomic modules is finite dimensional.

An important question left open in [11] is whether the KZ functor still essentially surjects onto the category of finite dimensional Hecke modules when we restrict to a certain subcategory consisting of *regular* $\mathscr{H}$-modules. These are defined in [12, Definition 1.5] to be those $\mathscr{O}$-coherent $\mathscr{H}$-modules for which each irreducible composition factor has regular singularities when viewed as an equivariant local system on a dense open set of its support. Let us note that in the rational case, by [12, Proposition 1.3] the category of $\mathscr{O}$-coherent regular H_c-modules is equal to the GGOR category $\mathscr{O}$. The approach we take to answering this question involves proving that the pushforward of an irreducible regular module under an open embedding is regular.

2.3. **An example.** As an example of a situation in which global Cherednik algebras appear naturally is the study of Feigin and Hakobyan of the *Dunkl angular momenta algebra* and the associated angular rational quantum Calogero-Moser superintegrable system. In particular, let $\mathbb{C}^N$ afford the permutation representation of the symmetric group S_N, and let $Q \subset \mathbb{P}(\mathbb{C}^N)$ be the codimension 1 subvariety of projective space defined by $\sum_{i=1}^N x_i^2 = 0$. Then Q is a smooth S_N-variety. In ongoing work with Feigin, we study the algebra $\mathscr{H}_{c,\omega}(Q, S_N)$, in particular, realizing it at a quotient of the Dunkl angular momenta algebra by a central character.

3. Further directions

3.1. **Gluing description.** We would like to devlop the theory of nearby and vanishing cycles for Cherednik modules. Ideally, one would like refine the support filtration on the category of regular holonomic $\mathscr{H}$-modules, using Riemann-Hilbert along the strata and a quiver-theoretic interpretation of the categories $\mathscr{O}$, to give a description in terms of gluings of quiver module categories. It is not difficult to produce directly a combinatorical description (which depends on the parameters) of the gluing datum for category of regular holonomic H_c-modules in rank 1 in terms of a subspace of the representation space of a specified quiver.

3.2. **Quantum Hamiltonian reduction.** In [5], Finkelberg and Ginzburg obtain the spherical subalgebra of the sheaf of Cherednik algebras on the Nth power of an algebraic curves with associated action of S_N as a quantum Hamiltonian reduction of the sheaf of (twisted) differential operators on a representation scheme associated to the curve. An interesting extension of our work on holonomic Cherednik modules is to show that the functor of Hamiltonian reduction here preserves holonomicity and regularity. This would even be interesting in the case where the curve is $\mathbb{C}^\times$. In this case, the Hamiltonian reduction functor takes the category of equivariant holonomic $\mathscr{D}$-modules on $GL_N(\mathbb{C}) \times \mathbb{C}^N$ to the representation category of the spherical trigonometric Cherednik algebra of type $\mathbf{A}_{N-1}$.

We generalize the above problem, in the rational case, in light of a result of [6] saying that the spherical cyclotomic rational Cherednik algebra is obtained as a certain quantized quiver variety: prove that the quantum Hamiltonian reduction functor from gauge-equivariant $\mathscr{D}$-modules on the quiver representation space preserves holonomicity and regularity. We should note that this example is also an instance of the case of [8, Theorem 1.3 (ii)]. Thus we may ask the question also more generally in the case of an arbitrary reductive algebraic group G acting by linear symplectomorphisms on a symplectic vector space V. We ask that the reduction functor from G-equivariant $\mathscr{D}(V)$-modules preserves holonomicity and regularity. We may have to be careful, however, as there is currenly only a working theory of holonomic modules on $\mathscr{D}(V)///G$ when we know that each symplectic leaf of the (classical) Hamiltonian reduction $V///G$ has finite algebraic fundamental group.

3.3. **Feigin-Hakobyan.** Using [2, Example 2.21] we plan to relate a generalization of the algebra of Feigin-Hakobyan algebra in rank $N = 3$, for finite subgroups of $SO_3(\mathbb{C})$, to rank 1 symplectic reflection algebras. What happens for other projective curves?

Another generalization of our work with Feigin: we would like to study Cherednik algebras $\mathscr{H}_{c,\omega}(X, W)$, where X is a hypersurface $\mathbb{P}(V)$ defined by $f = 0$, where V is a vector space having a linear action of W, and $f \in \mathbb{C}[V]^W$.

Finally, we would like to study the structure of the centre of the FH algebra in the classical case (i.e., for $t = 0$).

3.4. **Gauss-Manin connection.** We will study the direct image of Cherednik modules on a W-variety X under the map to a point. The direct image should be defined in the derived category when the twisting ω is zero. For example, in the affine space case, the pushforward of M is $M \otimes_W^L \mathscr{O}_X$, which, for $c = 0$ and $M = \mathscr{O}_X$, gives $H^*(X)^W$.

There are many questions to ask. What happens for arbitrary parameter c? Can one describe the answer in geometric terms (e.g., in terms of the geometry of reflection hypersurfaces, etc.)? What happens if we have a map $X \to Y$, where the action of W on Y is trivial? Do we get a Gauss-Manin connection on the bundle obtained by taking the direct image along fibers, and if so, what is the geometric description of this connection?

3.5. **Composition series.** In the case of the rational Cherendnik algebra, what is the composition series of the pushforward of a local systems (say, coming from Hecke algebra) on the regular locus? This is related to the distribution of roots of its b-function.

References

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