THE GLOBAL SECTIONS OF THE SHEAF OF CHEREDNIK ALGEBRAS OF A SMOOTH QUADRIC

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Abstract. We obtain the global sections algebra of the sheaf of Cherednik algebras $\mathscr{H}_{c,\omega}(Q,W)$ on Q, where $Q \subset \mathbb{P}^{N-1}$ is a smooth quadric hypersurface and W is a Coxeter group of rank at most N, as a quotient of the Dunkl angular momentum algebra $H_c^{\mathfrak{so}(N)}(W)$ by a central character. In the case $N = 3$ we relate these algebras to rank 2 symplectic reflection algebras.

1. Introduction

1.1. **Sheaves of Cherednik algebras.** Suppose X is a smooth variety with the action of a finite group W for which the quotient variety X/W exists. Etingof in a 2004 preprint [5] has defined a sheaf of Cherednik algebras $\mathscr{H}_{c,\omega}(X,W)$ on X in the W-equivariant topology (alternatively, as a sheaf on X/W) depending on parameters c, ω which we now explain.[1]

Define the set $\mathscr{S}(X) = \mathscr{S}(X,W)$ of *reflections* of X to be the set of pairs (w,Z) such that $w \in W$ and Z is an irreducible component of X^w having codimension 1 in X. Let $c : \mathscr{S}(X) \to \mathbb{C}$ be a W-invariant function. Take ω to be an element of $\mathbb{H}^2(X, \Omega_X^{\geq 1})^W$, where $\Omega_X^{\geq 1}$ is the two-term subcomplex $\Omega_X^1 \to (\Omega_X^2)^{\mathrm{cl}}$, concentrated in degrees 1 and 2, of the algebraic De Rham complex and where $(\Omega_X^2)^{\mathrm{cl}}$ denotes the subsheaf of closed forms in Ω_X^2. In the notation of [5], the sheaf of Cherednik algebras $\mathscr{H}_{c,\omega}(X,W)$ is written as $H_{1,c,\omega}(X,W)$ (i.e., $t = 1$).

In this paper, we write $H_{c,\omega}(X,W)$ to refer to the algebra of global sections of $\mathscr{H}_{c,\omega}(X,W)$ over X. If X is an affine variety, there is no harm in conflating the two, but our current interest is principally the case where X is projective.

Finally, there is also a "modified" sheaf of Cherednik algebras, written $\mathscr{H}_{c,\eta,\omega}(X,W)$, where η is a W-invariant $\mathbb{C}$-valued function on the

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[1] We cite an updated version which will soon appear in a volume dedicated to V. I. Arnol'd, though all references made herein follow the numbering from the 2004 arXiv version: v3.

set of hypersurfaces Z such that $(Z,s)\in S$ for some $s\in W$. We have $\mathscr{H}_{c,0,\omega}(X,W)=\mathscr{H}_{c,\omega}(X,W)$.

For convenience, we recall the definition of the sheaf $\mathscr{H}_{c,\eta,\omega}(X,W)$. Sheaves of twisted differential operators are classified by the space $\mathbb{H}^2(X,\Omega_X^{\geq 1})$, see section 2 of [1]. Let us write $\mathscr{D}_X^{\omega}$ to refer to the sheaf of twisted differential operators corresponding to ω. If we write $X'=X\setminus\bigcup_{(w,Z)\in\mathscr{S}(X)}Z$ and $j:X'\to X$ for the inclusion, then $\mathscr{H}_{c,\eta,\omega}(X,W)$ is defined as a subalgebra of the sheaf $j_*j^*(\mathscr{D}_X^{\omega}\rtimes\mathbb{C}W)$ generated locally by $\mathscr{O}_X$, $\mathbb{C}W$, and Dunkl operators D_y associated to vector fields y. Let $U\subset X$ be a W-stable affine open set. We may now define D_y locally, for $y\in\Gamma(U,TX)$. Let $\mathbb{L}_y\in\mathscr{D}_X^{\omega}(U)$ be the twisted Lie derivative corresponding to y. For every $(w,Z)\in\mathscr{S}(X)$, let $f_Z\in\Gamma(U,\mathscr{O}_X(Z))$ be a function whose residue map at Z agrees with y once both are restricted to the normal bundle of Z in X (as in [5, Definition 2.7]). Finally, let $\lambda_{(w,Z)}$ be the nontrivial eigenvalue of w on the conormal bundle to Z. Then on U we have

$$D_y=\mathbb{L}_y+\sum_{(w,Z)\in\mathscr{S}(X)}f_Z\left(\frac{2c(w,Z)}{1-\lambda_{(w,Z)}}(w-1)+\eta(Z)\right). \tag{1.1}$$

1.2. **Rational Cherednik algebra.** If $X=V$ is a vector space and W acts linearly on V, we say simply that an element of $s\in W$, $s\neq 1$, is a *reflection* if it fixes pointwise a codimension 1 hyperplane of V. Notice that the set $\mathscr{S}\subset W$ of reflections in this sense is in natural correspondence with the set $\mathscr{S}(V)$ of the previous paragraph. Given a function $c:\mathscr{S}\to\mathbb{C}$ which is constant on W-conjugacy classes in $\mathscr{S}$ we have the rational Cherednik algebra $H_c(V,W):=H_{1,c}(V,W)$ defined as follows. For $s\in\mathscr{S}$ choose eigenvectors $\alpha_s\in V^*$ and $\alpha_s^\vee\in V$ for the action of s, both with eigenvalue different from 1, and normalized so that $\langle\alpha_s,\alpha_s^\vee\rangle=2$ where $\langle\cdot,\cdot\rangle$ is the natural pairing between V^* and V. The rational Cherednik algebra $H_c(W,V)$ is defined (see [7]) to be the quotient of the smash-product algebra $T(V\oplus V^*)\rtimes\mathbb{C}W$ (here $T(V)$ denotes the tensor algebra of V) by relations of the following form:

$$[x,x']=0,\quad [y,y']=0,\quad [y,x]=\langle x,y\rangle-\sum_{s\in\mathscr{S}}c(s)\langle\alpha_s,y\rangle\langle x,\alpha_s^\vee\rangle s,$$

where $x,x'\in V^*,y,y'\in V$.

1.3. **A result of Bellamy-Martino.** Let W be a finite group acting linearly on $V=\mathbb{C}^N$. There is an induced action of W on $\mathbb{P}(V)$, and Lemma 5.4.1 of [2] tells us how to compute the global sections algebra $H_{c,\omega}(\mathbb{P}(V),W)$. We recall it now.

Let $H_c(V, W)$ be a rational Cherednik algebra. If $w \in \mathscr{S}$ is a reflection which fixes the hyperplane $H \subset V$, then of course $(w, \mathbb{P}(H)) \in \mathscr{S}(\mathbb{P}(V))$. Define $c' : \mathscr{S}(\mathbb{P}(V)) \to \mathbb{C}$ by

$$c'(w, \mathbb{P}(H)) = c(w)$$

and $c'(w, Z) = 0$ for all other $(w, Z) \in \mathscr{S}(\mathbb{P}(V))$. For example, in the case when $N = 2$ and $W = S_2$, the nontrivial element $s \in S_2$ has two fixed points on $\mathbb{P}^1$, only one of which comes from a reflection hyperplane of $\mathbb{C}^2$. However, for S_N with $N > 2$ all reflections of $\mathbb{P}(V)$ take the form $(w, \mathbb{P}(H))$. In general, we might even have reflections $(w, Z) \in \mathscr{S}(\mathbb{P}(V))$ where w is not a reflection on V, as it would be enough for αw to be a reflection for some $\alpha \in \mathbb{C}^\times$.

Recall that the rational Cherednik algebra $H_c(V, W)$ is graded by putting $\deg(x) = 1, \deg(y) = -1$, and $\deg(\mathbb{C}W) = 0$. In fact this grading is internal, defined by the element

$$\mathbf{h} = \sum_{i=1}^{N} x_i y_i - \sum_{s \in \mathscr{S}} \frac{2c(s)}{1 - \lambda_s} s.$$

Here λ_s is the nontrivial eigenvalue for s on $\mathfrak{h}^*$, x_i is a basis of V^*, and y_i the dual basis of V. Write $H_c(V, W)_m$ to refer to the mth graded piece of $H_c(V, W)$.

Given a line bundle $\mathscr{L}$ on $\mathbb{P}(V)$, let $\omega_{\mathscr{L}} \in \mathbb{H}^2(\mathbb{P}(V), \Omega^{\geq 1}_{\mathbb{P}(V)})$ be given by its curvature. Finally, let us identify $\mathbb{H}^2(\mathbb{P}(V), \Omega^{\geq 1}_{\mathbb{P}(V)})$ with $\mathbb{C}$ in such a way that $\omega_{\mathscr{O}(n)} = n$.

Proposition 1.1 (Lemma 5.4.1 of [2]). *Fix a parameter c for the rational Cherednik algebra $H_c(V, W)$, and define c' as above. Then we have*

$$H_{c',\omega}(\mathbb{P}(V), W) = H_c(V, W)_0 / (\mathbf{h} + \sum_{s \in \mathscr{S}} \frac{2c(s)}{1 - \lambda_s} - \omega).$$

For arbitrary parameter c for $(\mathbb{P}(V), W)$, a computation of $H_{c,\omega}(\mathbb{P}(V), W)$ is contained in [5, Example 2.20]. It can be deduced from the above theorem by passing to a possibly larger group than W.

1.4. **The quadric hypersurface.** Now we fix $W = S_N$ with $N > 2$, let $V = \mathbb{C}^N$ afford the permutation action of W, and let c be a parameter for the rational Cherednik algebra $H_c(V, W)$ (in this case, $c \in \mathbb{C}$ since all reflections are conjugate). Let $Q \subset \mathbb{P}(V)$ be the smooth connected quadric hypersurface defined by $\sum_{i=1}^{N} {x_i}^2 = 0$. Now Q is stable under the action of W, and since Q is not contained in any reflection hyperplane of $\mathbb{P}(V)$ it is immediate that Q is transverse to each of them. Thus any W-invariant function $c' : \mathscr{S}(\mathbb{P}(V)) \to \mathbb{C}$

restricts to a function $c'|_Q : \mathscr{S}(Q) \to \mathbb{C}$ via $c'|_Q(w, Z \cap Q) = c'(w, Z)$ and $c'|_Q(w, Z) = 0$ for all other $(w, Z) \in \mathscr{S}(Q)$. We note that in general there may be reflections of Q that are not of the form $(w, Z \cap Q)$, though such reflections do not occur if $W \subset GL(V)$ is generated by its reflections $\mathscr{S}$ (as will always be the case in this paper).

We give for a way to compute $H_{c'|_Q,\omega|_Q}(Q, W)$ in the spirit of Proposition 1.1.

Theorem 1.2. *Let $W = S_N$ for $N > 2$ and fix a parameter c for the rational Cherednik algebra $H_c(V, W)$. Define $c'(w, \mathbb{P}(H)) = c(w)$ for $H \subset V$ a reflection hyperplane given by $x_i = x_j$. Then we have an isomorphism between $H_{c'|_Q,\omega|_Q}(Q, W)$ and $H_c^{\mathfrak{so}(N)} / (H_\Omega + a)$, with $a = (\omega - \frac{cN(N-1)}{2})(\omega - \frac{cN(N-1)}{2} + N - 2)$.*

Here $H_c^{\mathfrak{so}(N)} = H_c^{\mathfrak{so}(N)}(S_N)$ is the Dunkl angular momentum algebra introduced by Hakobyan and one of the authors in [8], and $H_\Omega \in H_c^{\mathfrak{so}(N)}$ is an element which generates its center (definitions of both are recalled in §2.2 below).

The structure of this paper is as follows. Section 2 will be devoted to proving main Theorem 1.2. Along the way we observe in Propositions 2.1, 2.2 that the cotangent bundle to Q is a resolution of a particular nilpotent orbit in $\mathfrak{so}(N)$. In section 3, we state a more general version of Theorem 1.2, where S_N is now replaced by any finite Coxeter group W, whose proof is identical. Our next result of interest is Theorem 3.3, the statement of which was first asserted by Etingof in [5] without specification of parameters. It relates (most) symplectic reflection algebras (SRAs) in rank 2 to global sections algebras for sheaves of Cherednik algebras on $\mathbb{P}^1$. This relation provides an isomorphism between a certain "partialy spherical" subalgebra of all symplectic reflection algebras for finite subgroups of $SL_2(\mathbb{C})$ which are not cyclic of odd order and the global sections algebra of a Cherednik sheaf for a corresponding finite group of automorphisms of $\mathbb{P}^1$. This allows us to give an isomorphism of these partialy spherical SRAs with quotients of Dunkl angular momentum algebras.

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2. Proof of the main theorem

2.1. **Geometric preliminaries.** In this section we study the closure of a certain nilpotent orbit $\mathbb{O}$ of $\mathfrak{so}_N$. The results here will be used in the proof of Theorem 1.2.

Fix a symmetric nondegenerate form on V preserved by $\mathfrak{so}_N$. These are unique up to scale, and in coordinates the form is given by $\sum_{i=1}^N {x_i}^2$. Define $\mathbb{O} \subset \mathfrak{so}_N$ to be the space of rank 2 nilpotent matrices A such that the form is nonzero on the image of A in V. It is easy to see that the partition-type classification (e.g., [4, Ch. 5]) of $\mathbb{O}$ is $(3, 1, \ldots, 1)$. Indeed, this claim follows from Propositions 5.2.5 and 5.2.8 of [4]. We note that $\overline{\mathbb{O}}$ has dimension $2N-4$, while the minimal nilpotent orbit of $\mathfrak{so}_N$ (*orbits* if $N = 4$) has dimension $2N - 6$ (see [12]).

For any projective variety $X \subset \mathbb{P}^r$, let C_X denote the cone over X in $\mathbb{A}^{r+1}$, and let $C_X^\circ = C_X \setminus \{0\} \subset \mathbb{A}^{r+1} \setminus \{0\}$. Let $G(2, N)$ denote the Grassmannian of 2-planes in V. Let x_{ij} (for $1 \le i < j \le N$) be the Plücker coordinates for $G(2, N)$, viewed as elements of $\mathbb{C}[C_{G(2,N)}]$ under the natural isomorphism of the later with the homogeneous coordinate ring of $G(2, N)$.

Proposition 2.1. *The orbit closure $\overline{\mathbb{O}}$ is the subvariety of $C_{G(2,N)}$ given by $\sum_{i<j} x_{ij}^2 = 0$.*

Proof. Observe that the orbit closure $\overline{\mathbb{O}} \subset \mathfrak{so}_N$ is the space of rank at most 2 nilpotent matrices in $\mathfrak{so}_N$. We must give a map $\overline{\mathbb{O}} \to C_{G(2,N)}$.

Use the form to make the SO_N-equivariant identification $\mathfrak{so}_N \simeq \Lambda^2 V$. A rank 2 matrix in $\mathfrak{so}_N$, then, can be written as $A = v \otimes w - w \otimes v$ for $v, w \in V$ linearly independent, nonzero vectors (and well-defined up to the action of SL_2). On the other hand, as varieties, $C_{G(2,N)}$ is isomorphic to the space of ordered pairs of linearly independent vectors (v, w) modulo the action of SL_2 (just as $G(2, N)$ can be thought of as the space of pairs of linearly independent vectors (v, w) modulo the action of GL_2) and zero.

Define a map of varieties $\overline{\mathbb{O}} \to C_{G(2,N)}$ by $v\otimes w-w\otimes v \mapsto (v, w)$. The map is clearly injective; we claim this is the desired closed embedding. For a rank 2 matrix $A = v \otimes w - w \otimes v \in \mathfrak{so}_N$ is nilpotent if and only if the form is degenerate when restricted to the span of $\{v, w\}$. Also, the point $(v, w) \in C_{G(2,N)}$ satisfies the condition $\sum_{i<j} x_{ij}^2 = 0$ if and only if the form is degenerate. This concludes the proof. □

Recall now that $Q \subset \mathbb{P}(V)$ is the projectivization of the space of isotropic vectors in V. Equivalently, the quadric Q is a quotient of the space of isotropic vectors in V° by $\mathbb{C}^\times$. At the image $\overline{v}$ of a point $v \in V$ with $v^2 = 0$ we may represent the cotangent space as

$$T^*_{\overline{v}} Q = v^\perp / \mathbb{C}v,$$

with identifications $T^*_{\overline{v}}Q \to T^*_{\overline{\lambda v}}Q$ for $\lambda \in \mathbb{C}^\times$ given by multiplication by λ^{-1}. Now define the map $p : T^*Q \to \overline{\mathbb{O}}$ by

$$w \mapsto v \otimes w - w \otimes v$$

for any $w \in T^*_{\overline{v}}Q$. It follows from the proof of the previous Proposition that this map takes values inside of $\overline{\mathbb{O}}$.

Proposition 2.2. (cf. [9]) *The map p is projective and birational.*

Proof. To see that p is birational, it is enough to observe that the subset of T^*Q represented by elements (v, w), $w \in T^*_{\overline{v}}Q$, such that w is not isotropic, maps isomorphically onto $\mathbb{O}$. Indeed, given an element $A \in \mathbb{O}$ the corresponding $\overline{v}$ is the kernel of the bilinear form on the plane corresponding to A.

Now to show that the map p is projective, we consider the product $T^*Q \to \overline{\mathbb{O}} \times Q$ of p with the bundle projection. We have only to show this map is a closed embedding because Q is projective. But in fact, it is easy to see that the composition

$$T^*Q \to \overline{\mathbb{O}} \times Q \to \mathfrak{so}_N \times Q$$

is a vector subbundle over Q. □

2.2. **Construction of the map.** Let C_Q denote the cone over Q in V. Let V° be $V \setminus \{0\}$, and let $C_Q^\circ = C_Q \cap V^\circ$. As in subsection 2.1, c restricts to a function $c|_Q$ on the reflections of C_Q°. Now $C_Q^\circ \subset V^\circ$ is a smooth closed subvariety so by section 4.3.2 of [11], $\mathscr{H}_{c|_Q}(C_Q^\circ, W)$ is naturally a subsheaf of $\mathscr{H}_c(V^\circ, W)/\mathscr{I}_{C_Q^\circ}\mathscr{H}_c(V^\circ, W)$ where $\mathscr{I}_{C_Q^\circ}$ is the ideal sheaf of C_Q° in V°. Upon taking global sections, this inclusion becomes $H_{c|_Q}(C_Q^\circ, W) \subset H_c(V, W)/IH_c(V, W)$ where $I = (\sum_{i=1}^N {x_i}^2)$, as can be seen from the elements given in the next paragraph. Since $IH_c(V, W)$ is a homogeneous ideal, the right module $H_c(V, W)/IH_c(V, W)$ is again graded.

The subalgebra $H_{c|_Q}(C_Q^\circ, W)$ is generated over $\mathbb{C}[C_Q^\circ] = \mathbb{C}[x_1, \dots, x_N]/I$ by the residues modulo I of elements of the form $x_i y_j - x_j y_i$ and $\mathbf{h}$, as well as by $\mathbb{C}W$. As these generators are all homogeneous of degree 0, the algebra $H_{c|_Q}(C_Q^\circ, W)$ also has a grading inherited from $H_c(V, W)$.

The Dunkl angular momentum algebra $H_c^{\mathfrak{so}(N)}$ is defined in [8] to be the subalgebra of the rational Cherednik algebra $H_c(V, W)$ generated over $\mathbb{C}W$ by elements

$$M_{ij} = x_i y_j - x_j y_i.$$

Hence we see that the composite map $H_c^{\mathfrak{so}(N)} \to H_c(V, W)/IH_c(V, W)$ in fact takes values in the degree zero part of $H_{c|_Q}(C_Q^\circ, W)$.

Finally, we construct a map $H_{c|_Q}(C_Q^\circ, W)_0 \to H_{c'|_Q,\omega|_Q}(Q,W)$. Let $\pi : C_Q^\circ \to Q$ be the quotient map (a principal $\mathbb{C}^\times$-bundle). By Proposition 4.3.2 of [2], we have a map of sheaves of algebras on Q given by

$$(\pi_* \mathscr{H}_{c|_Q}(C_Q^\circ, W))^{\mathbb{C}^\times} \to \mathscr{H}_{c'|_Q,\omega|_Q}(Q,W).$$

(If c' is defined as in the previous section, then $c'|_Q$ is such that π is melys and $c|_Q$ is its pullback.) Taking global sections we get a map

$$\varphi : H_{c|_Q}(C_Q^\circ, W)_0 = H_{c|_Q}(C_Q^\circ, W)^{\mathbb{C}^\times} \to H_{c'|_Q,\omega|_Q}(Q,W).$$

We claim that $\mathbf{h} + \frac{cN(N-1)}{2} - \omega$ is in the kernel of this map; this can immediately be seen from [2, Proposition 4.3.2] by restricting π to a principal $\mathbb{C}^\times$-bundle over an affine open set of Q, and from the fact that the map β there is compatible with restriction.

Let

$$\psi : H_c^{\mathfrak{so}(N)} \to H_{c|_Q}(C_Q^\circ, W)_0$$

be the natural inclusion. Define the composite map

$$\Psi = \varphi \circ \psi : H_c^{\mathfrak{so}(N)} \to H_{c'|_Q,\omega|_Q}(Q,W).$$

Lemma 2.3. *Let H_Ω be the the angular Calogero–Moser Hamiltonian:*

$$H_\Omega = \sum_{i<j} M_{ij}^2 - S(S-N+2),$$

where $S = \sum_{i<j} cs_{ij}$, and s_{ij} denote the reflections in S_N. Then

$$\Psi\left(H_\Omega + (\omega - \frac{cN(N-1)}{2})(\omega - \frac{cN(N-1)}{2} + N - 2)\right) = 0.$$

Proof. It follows from equation (2.14) of [8] that

$$\Psi(\sum_{i<j} M_{ij}^2) = -\varphi(\mathbf{h}+S)^2 + (2S-N+2)\varphi(\mathbf{h}+S).$$

Since we also have $\varphi(\mathbf{h}+S) = -c\frac{N(N-1)}{2} + \omega + S$, the statement follows. $\square$

2.3. **Associated graded algebras and isomorphism.** Finally, we must check that modulo this central character, the map Ψ is an isomorphism. Now both of these algebras have compatible order filtrations, and we know what the associated graded of each is.

By [8, Theorem 4] $\mathrm{gr} H_c^{\mathfrak{so}(N)}$ is the smash product of W with homogeneous coordinate ring for the Grassmannian $G(2,N)$.

Let $\rho : T^*Q \to Q$ be the canonical projection. By the usual description of T^*Q as the Hamiltonian reduction of $T^*C_Q^\circ$ with respect to the induced $\mathbb{C}^\times$-action, we can calculate

$$\mathrm{gr}H_{c'|_Q,\omega|_Q}(Q,W) = \Gamma(Q, \rho_*\mathscr{O}_{T^*Q} \rtimes \mathbb{C}W) = \mathbb{C}[T^*Q] \rtimes \mathbb{C}W$$

as $\mathbb{C}[T^*C_Q^\circ]_0/(\sum_i x_i y_i) \rtimes \mathbb{C}W$. For

$$\mathrm{gr}\mathscr{H}_{c'|_Q,\omega|_Q}(Q,W) = \rho_*\mathscr{O}_{T^*Q} \rtimes \mathbb{C}W$$

by [5, Theorem 2.11], and taking associated graded is seen to commute with taking global sections. To see this, consider the short exact sequences

$$0 \to \mathscr{F}^{i-1}(Q,W) \to \mathscr{F}^i(Q,W) \to (\mathrm{Sym}^i\Theta_Q) \rtimes \mathbb{C}W \to 0,$$

where $\mathscr{F}^i(Q,W)$ denotes the order filtration on $\mathscr{H}_{c'|_Q,\omega|_Q}(Q,W)$ and Θ_Q is the tangent sheaf, and show by induction that $\mathscr{F}^i(Q,W)$ is acyclic. The third term in each sequence has vanishing higher cohomology by [3, §A2].

We have $\mathbb{C}[C_Q^\circ] = \mathbb{C}[x_1, \dots, x_N]/(\sum_i {x_i}^2)$ and $\mathbb{C}[T^*C_Q^\circ]$ is the subalgebra of $\mathbb{C}[C_Q^\circ][y_1, \dots, y_N]$ generated over $\mathbb{C}[C_Q^\circ]$ by elements $x_i y_j - x_j y_i$ and $\sum_{i=1}^N x_i y_i$. It is apparent from this description that the associated graded map is surjective.

Now we turn to computing the kernel of

$$\mathrm{gr}\Psi : \mathbb{C}[C_{G(2,N)}] \rtimes \mathbb{C}W \to \mathbb{C}[T^*Q] \rtimes \mathbb{C}W.$$

In fact, it is clearly enough to compute the kernel of the restriction

$$\mathbb{C}[C_{G(2,N)}] \to \mathbb{C}[T^*Q].$$

Now by Proposition 2.2 we know that T^*Q is a symplectic resolution of singularities of a certain nilpotent orbit closure $\overline{\mathbb{O}} \subset \mathfrak{so}_N(\mathbb{C})$, and since $\overline{\mathbb{O}}$ is normal ([10, §5]) we have $\mathbb{C}[T^*Q] = \mathbb{C}[\overline{\mathbb{O}}]$. Now, we have a surjective map from the coordinate ring of an integral scheme (cone over the Grassmannian) to the coordinate ring of an integral scheme (a nilpotent orbit closure) of one dimension less. Thus it suffices to note that the ideal of $\mathbb{C}[C_{G(2,N)}]$ generated by $\mathrm{gr}H_\Omega$ (i.e., the sum of squares of the Plücker coordinates) is prime. For the cone in $\mathbb{A}^{\binom{N}{2}} = \mathrm{Spec}\,\mathbb{C}[x_{ij} | 1 \le i < j \le N]$ given by $\sum_{i<j} x_{ij}^2 = 0$ is integral because $\sum_{i<j} x_{ij}^2$ is an irreducible polynomial.

Lemma 2.4. *The intersection of $C_{G(2,N)}$ with the cone in $\mathbb{A}^{\binom{N}{2}} = \mathrm{Spec}\,\mathbb{C}[x_{ij} | 1 \le i < j \le N]$ given by $\sum_{i<j} x_{ij}^2 = 0$ is an integral scheme.*

Proof. In Proposition 2.1, we have already shown that the intersection is equal to $\overline{\mathbb{O}}$ *as a variety*, which implies that the scheme-theoretic intersection is irreducible. Thus we have only to show the intersection is reduced.

Since the cone over an reduced scheme is reduced, it will suffice to consider the intersection between the corresponding quadric hypersurface and $G(2,N)$ inside of $\mathbb{P}^{\binom{N}{2}-1}$. This intersection defines a hypersurface H in $G(2,N)$. Now the Grassmannian $G(2,N)$ has an open cover by the sets U_{ij} defined as $x_{ij}=1$, and each $U_{ij}\simeq \mathbb{A}^{2(N-2)}$. As such, $\mathbb{C}[U_{ij}]$ is a polynomial algebra in the coordinates x_{kl} with $|\{k,l\}\cap\{i,j\}|=1$. On each chart U_{ij}, the hypersurface H is the variety defined by the polynomial

$$1+\sum_{|\{k,l\}\cap\{i,j\}|=1} x_{kl}^2+\sum_{|\{k,l\}\cap\{i,j\}|=0}(x_{ik}x_{jl}-x_{il}x_{jk})^2,$$

and this polynomial is easily seen to be square-free, so H is integral. $\square$

By Lemma 2.4, the ideal $(\sum_{i<j} x_{ij}^2)$ remains prime in $\mathbb{C}[C_{G(2,N)}]$. This concludes the proof of Theorem 1.2.

3. Coxeter group generalizations

3.1. In this section, we suppose W is a real reflection group, thus acting faithfully, on the vector space $V=\mathbb{C}^N$. For a choice of parameter c, we have a rational Cherednik algebra $H_c(V,W)$. Choose a basis x_i of V^*, and y_i a dual basis of V. We may define the Dunkl angular momenta algebra $H_c^{\mathfrak{so}(N)}(W)$ to be the subalgebra of the rational Cherednik algebra $H_c(V,W)$ generated over $\mathbb{C}W$ by elements $M_{ij}=x_iy_j-x_jy_i$. The key results of [8] hold for these algebras as well, see section 8 of that paper: we have $\mathrm{gr}H_c^{\mathfrak{so}(N)}(W)=\mathbb{C}[C_{G(2,N)}]\rtimes\mathbb{C}W$, and there is a central element $H_\Omega\in H_c^{\mathfrak{so}(N)}(W)$ defined as

$$H_\Omega=\sum_{i<j} M_{ij}^2-S(S-N+2),$$

where $S=\sum_{s\in\mathscr{S}} c(s)s$, and again $\mathscr{S}$ denotes the set of reflections of W. The same argument used to prove Theorem 1.2 in fact gives us the following.

Theorem 3.1. *Fix a parameter c for the rational Cherednik algebra $H_c(V,W)$. Define $c'(w,\mathbb{P}(H))=c(w)$ for $H\subset V$ a reflection hyperplane for the action of W, and $c'(w,Z)=0$ otherwise. Then we have isomorphism of algebras*

$$H_{c'|_Q,\omega|_Q}(Q,W)\cong H_c^{\mathfrak{so}(N)}(W)/\left(H_\Omega+a\right),$$

where $a = (\omega - \sum_{s\in\mathscr{S}} c(s))(\omega + N - 2 - \sum_{s\in\mathscr{S}} c(s))$.

Remark 3.2. It would be interesting to see whether there are analogues of Dunkl angular momentum algebras such that Theorem 3.1 can be generalised for the case of some W-invariant smooth (hyper)surfaces of degree bigger than 2 for some (complex) reflection groups W.

3.2. Relation of $H_c^{\mathfrak{so}(N)}(W)$ to symplectic reflection algebra. Let us specialize to the case $N = 3$, and suppose still that W is a real reflection group. According to the classification of Coxeter groups, W is of type $A_3, B_3, H_3, A_1 \times I_m, I_m$, or A_1, where $m \geq 2$ with I_2 standing for $A_1 \times A_1$.

By the well-known genus-degree formula, the quadric is $Q = \mathbb{P}^1$, and our identifications $\mathbb{H}^2(\mathbb{P}^n, \Omega^{\geq 1}_{\mathbb{P}^n}) \simeq \mathbb{C}$ (§1.3) are such that if $\omega = \omega_{\mathbb{P}^3} = n$ then $\omega|_Q = 2n$ because Q has degree 2. Now the group W acts on Q by algebraic automorphisms, giving us a map $W \to PGL_2(\mathbb{C}) = PSL_2(\mathbb{C})$. Note that the kernel K of this map is $W \cap \{\pm \mathrm{Id}\}$, the intersection taken in $GL_3(\mathbb{C})$. Let $\overline{W} = W/K$ be the image of W in $PSL_2(\mathbb{C})$.

If K is nontrivial, it is easy to see that we have a natural isomorphism

$$H_{c'|_Q,\omega|_Q}(Q, W)/((-\mathrm{Id}) - 1) \simeq H_{c'',\omega|_Q}(Q, \overline{W}), \tag{3.1}$$

where $c''(\overline{s}, Z) = \sum_{k\in K} c'|_Q(sk, Z)$. Recall that by definition, we have that $c'|_Q(sk, Z) \neq 0$ only if sk is a reflection on V. Since $V = \mathbb{C}^3$, s and $-s$ cannot both be reflections of V. Thus $c''(\overline{s}, Z) = c'|_Q(s, Z)$ if $s \in \mathscr{S}(V)$, and $c''(\overline{s}, Z) = c'|_Q(-s, Z)$ otherwise.

Now we may apply the result of Example 2.21 of [5] to express $H_{c'',\omega|_Q}(Q, \overline{W})$ as a "partially spherical" subalgebra of the rank 2 symplectic reflection algebra corresponding to the preimage of $\overline{W}$ under the map $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$. Combining this with Theorem 3.1 allows us to relate the quotient of $H_c^{\mathfrak{so}(N)}(W)$ by a central character to this partially spherical SRA.

We are unaware of a proof of the claim of [5, Example 2.21] in the literature, so it is this to which we must attend first. Let Γ be the preimage of $\overline{W} \subset PSL_2(\mathbb{C})$ in $SL_2(\mathbb{C}) = Sp_2(\mathbb{C})$. (For the next proposition, $\overline{W}$ is allowed to be any finite subgroup, that is, it does not necessarily come from a W at the beginning of the subsection.) The kernel of the map $\Gamma \to \overline{W}$ consists of the identity 1 and minus identity z. Let $p_z \in \mathbb{C}[\Gamma]$ be the idempotent $p_z = (1 + z)/2$. Let $\kappa \in \mathbb{C}[\Gamma]^\Gamma$, and let $A_\kappa = A_\kappa(\Gamma)$ be the SRA associated with $\Gamma \subset SL_2(\mathbb{C})$. A_κ is defined to be the quotient of the smash product $T(\mathbb{C}^2) \rtimes \mathbb{C}\Gamma$ by the ideal generated by $vu - uv - \kappa$ (where $T(\mathbb{C}^2)$ in the tensor algebra of the tautological representation of Γ, and v, u are a basis of $\mathbb{C}^2$). Finally, each $s \in \overline{W} \setminus \{1\}$ fixes two points of $\mathbb{P}^1$. Moreover, we have a bijection $\xi : \Gamma \setminus \{1, z\} \to \mathscr{S}(\mathbb{P}^1, \overline{W})$ given by $\xi : \gamma \mapsto (\overline{\gamma}, Y^+)$, where $\overline{\gamma}$ is the projection of γ to $\overline{W}$ and where Y^+ is the projectivization of the fixed line of γ on which it acts with eigenvalue λ_γ having positive imaginary part. The bijection ξ takes Γ-conjugacy classes to $\overline{W}$-conjugacy classes. We have $\lambda_{z\gamma} = -\overline{\lambda_\gamma}$.

Theorem 3.3. *Let $\overline{W} \subset PSL_2(\mathbb{C})$ be any finite subgroup, and let Γ be its preimage in $SL_2(\mathbb{C})$. Suppose κ is of the form*

$$\kappa = |\Gamma|^{-1} + \sum_{\gamma \in \Gamma \setminus \{1\}} \kappa_\gamma \cdot \gamma.$$

Then there is an isomorphism $p_z A_\kappa p_z \cong H_{c,\phi}(\mathbb{P}^1, \overline{W})$, with the Cherednik parameters given by

$$c(\xi(\gamma)) = -\frac{|\Gamma|}{4}((1 + \lambda_\gamma)\kappa_\gamma + (1 - \lambda_\gamma)\kappa_{z\gamma}), \tag{3.2}$$

$$\phi = -\frac{1 + \kappa_z|\Gamma|}{2} - \sum_{\gamma \in \Gamma \setminus \{1,z\}} \frac{2c(\xi(\gamma))}{(1 - \lambda_\gamma^2)(|\overline{W}_y| - 1)}, \tag{3.3}$$

where the point $y \in \mathbb{P}^1$ in the sum is given by $\xi(\gamma) = (\overline{\gamma}, y)$, and $\overline{W}_y$ denotes its stabilizer.

Proof. We use the map $\theta^{\mathrm{Dunkl}} : A_\kappa \to B$ of [6, Theorem 4.3.2] in the $n = 1$ case, where B denotes the target of that map.

Let

$$\psi = -\frac{1}{2}(1 + \kappa_z|\Gamma|). \tag{3.4}$$

Multiplying this map on both sides by the idempotent p_z, the map becomes an injection from

$$p_z A_\kappa p_z \to \Gamma(X', \mathscr{D}_{X'}^{\psi|_{X'}}) \rtimes \mathbb{C}\overline{W},$$

with $\mathscr{D}_{X'}^{\psi|_{X'}}$ denoting the sheaf of twisted differential operators on the $\overline{W}$-regular locus of $\mathbb{P}^1$:

$$X' = \mathbb{P}^1 \setminus \bigcup_{(s,y) \in \mathscr{S}(\mathbb{P}^1)} y.$$

To explain why this is so, first note that $p_z A_\kappa p_z$ lives in even degree in v and w, and so the image of θ^{Dunkl} is represented by 2×2 diagonal matrices. However, after taking the equalizer [6, §4.2] and passing to the spherical subalgebra, the subalgebra of $p_z B p_z$ generated by diagonal matrices with first diagonal term 0 is zero. Now following the construction of B given in [6, §3.1], we see that the subalgebra of p_zBp_z generated by diagonal matrices is equal to the subalgebra generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} p_z \left(\Gamma(X', \mathscr{D}_{X'}^{\psi|_{X'}}) \rtimes \mathbb{C}\Gamma \right) p_z,$$

and hence is isomorphic to $\Gamma(X', \mathscr{D}_{X'}^{\psi|_{X'}}) \rtimes \mathbb{C}\overline{W}$ (see [2, Proposition 7.3.2]).

Now to see that the image of $p_z\theta^{\mathrm{Dunkl}}p_z$ inside of $\Gamma(X', \mathscr{D}_{X'}^{\psi|_{X'}}) \rtimes \mathbb{C}\overline{W}$ is isomorphic to $H_{c,\phi}(\mathbb{P}^1, \overline{W})$, we work affine-locally on $\mathbb{P}^1$. The algebra $p_zA_\kappa p_z$ is generated by $p_zv^2p_z, p_zvup_z, p_zuvp_z, p_zu^2$, where v, u give a basis for $\mathbb{C}^2$. According to the definition of θ^{Dunkl}, the image is generated over $\mathbb{C}\overline{W}$ by the images of the four elements $p_zv^\vee D^vp_z$, $p_zv^\vee D^up_z$, $p_zu^\vee D^vp_z$, $p_zu^\vee D^up_z$ in $\Gamma(X', \mathscr{D}_{X'}^{\psi|_{X'}}) \rtimes \mathbb{C}\overline{W}$, where, for any $v \in \mathbb{C}^2$, we have

$$D^v = v + \frac{|\Gamma|}{2} \sum_{\gamma \in \Gamma \setminus \{1,z\}} \kappa_\gamma \frac{(\gamma v + v)^\vee}{\omega^\gamma} \gamma.$$

Here ω is the symplectic form on $\mathbb{C}^2$, $(v)^\vee = \iota_v\omega$ is a linear functional for any $v \in \mathbb{C}^2$, and ω^γ is the quadratic function given by $\omega^\gamma(v) = \omega(v, \gamma v)$. These four elements induce twisted differential operators on $\mathbb{P}^1$ with poles at Γ-fixed points. We must check that locally they induce "modified" Dunkl operators defined in equation (1.1), so they define global sections of the sheaf $\mathscr{H}_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$, with parameters c, ψ as in (3.2), (3.4) and $\eta(y) = \frac{1}{|\overline{W}_y|-1} \sum_{w \in \overline{W}_y} \frac{2c(w,y)}{1-\lambda_{(w,y)}}$. First we will check this in the case that $\overline{W}$, hence Γ, is cyclic, then we will show how this result implies the statement for any group $\overline{W}$. Finally, we will compute how the modification affects the global twisting parameter in the untwisted Cherednik algebra, to get (3.3).

Suppose now that Γ cyclic with eigenvectors v, u of $\mathbb{C}^2$, normalized so that $\omega(v, u) = 1$. For each $\gamma \in \Gamma \setminus \{1, z\}$, say v, u have corresponding eigenvalues $\overline{\lambda_\gamma}, \lambda_\gamma$ respectively. We check that $p_zv^\vee D^vp_z$ restricts to a Dunkl operator on the chart $U = \mathbb{P}^1 \setminus \mathbb{P}(\mathbb{C}v)$ of $\mathbb{P}^1$. Of course, $U \simeq \mathbb{A}^1$ is affine and $\overline{W}$-stable; let $x = -u^\vee/v^\vee$ be a coordinate of this chart, and $\mathbb{C}[U] = \mathbb{C}[x]$. Denote by 0 the point $\mathbb{P}(\mathbb{C}u)$ of $\mathbb{P}^1$.

We compute directly that the differential operator induced on U by $p_zv^\vee D^vp_z$ is

$$\partial_x + \sum_{\gamma \in \Gamma \setminus \{1,z\} : \xi(\gamma) = (\overline{\gamma}, 0)} \frac{2c(\xi(\gamma))}{1 - \lambda_\gamma^2} \frac{1}{x} \overline{\gamma}$$

(note that on U, $\partial_x = \mathbb{L}_{\partial_x}$ since $H^2(U, \mathbb{C}) = 0$). The other three generators are easy to calcluate on U, using the fact that the Euler element $-u^\vee \partial_v + v^\vee \partial_u$ acts on $\mathbb{P}^1$ by the constant ψ (as this is its image in $\mathscr{D}^\psi_{\mathbb{P}^1}(\mathbb{P}^1)$). We conclude that

$$p_z u^\vee D^v p_z, \quad p_z v^\vee D^u p_z, \quad p_z u^\vee D^u p_z,$$

differ by elements of $\mathbb{C}[U] \rtimes \mathbb{C}\overline{W}$ from

$$-x\partial_x, \quad -x\partial_x, \quad x^2\partial_x,$$

respectively. By definition, these are all elements of $H_{c,\eta,\psi}(U, \overline{W})$, a rational Cherednik algebra.

Now we turn to the situation of a general group $\overline{W}$. To show that the image of $p_z\theta^{\mathrm{Dunkl}}p_z$ consists of sections of $\mathscr{H}_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$, by the fact below it suffices to check that this is the case at formal neighborhoods of each point of $\mathbb{P}^1/\overline{W}$.

Lemma 3.4. *Let $f : \mathscr{F} \to \mathscr{G}$ be a map of quasicoherent sheaves on a variety, with $\mathscr{F}$ coherent. Then $f = 0$ if and only if the completion of the map, $\widehat{f}_q : \widehat{\mathscr{F}}_q \to \widehat{\mathscr{G}}_q$, at every point q, is equal to zero.*

Proof. The lemma follows easily from standard results of commutative algebra. $\square$

To apply the lemma to our case, we take $\mathscr{F}$ to be the subsheaf of $\mathscr{D}^{\psi|_{X'}}_{X'} \rtimes \mathbb{C}\overline{W}$ generated (over $\mathscr{O}_X$) by those global sections in the image of $p_z\theta^{\mathrm{Dunkl}}p_z$, and then take f to be the composition of the aforementioned inclusion with the quotient by $\mathscr{H}_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$.

Now since the desired containment,

$$p_z\theta^{\mathrm{Dunkl}}p_z \subset \mathscr{H}_{c,\eta,\psi}(\mathbb{P}^1, \overline{W}),$$

is trivial when restricted to X', we have only to consider the completion at points y with nontrivial stabilizer. But by [13, Proposition 2.6], the completion of the sheaf at $\overline{W} \cdot y$ can be written:

$$\widehat{\mathscr{H}_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})}_{\overline{W}\cdot y} \simeq \mathrm{Mat}_{[\overline{W},\overline{W}_y]}\left(\widehat{H_{c,\eta}(T_yX, \overline{W}_y)_0}\right).$$

There is a similar formula for the completion of the pushforward under the inclusion $j : X' \to \mathbb{P}^1$ of $\mathscr{D}^{\psi|_{X'}}_{X'} \rtimes \mathbb{C}\overline{W}$ (of which $\mathscr{H}_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$ is a subsheaf):

$$\widehat{\left(j_*\mathscr{D}^{\psi|_{X'}}_{X'}\right)}_{\overline{W}\cdot y} \rtimes \mathbb{C}\overline{W} \simeq \mathrm{Mat}_{[\overline{W},\overline{W}_y]}\left(\widehat{\mathscr{D}_{T_yX}(T_yX \setminus \{0\})_0} \rtimes \mathbb{C}\overline{W}_y\right). \tag{3.5}$$

These isomorphisms depend on the choice of a set T of left coset representatives for $\overline{W}/\overline{W}_y$. Now following the proof of [13, Proposition 2.6], we describe the image under the isomorphism (3.5) of the restriction to this formal neighborhood of the section of $\mathscr{D}_{X'}^{\psi|_{X'}} \rtimes \mathbb{C}\overline{W}$ represented by $p_z u^\vee D^v p_z$. The off-diagonals terms belong to

$$\widehat{\mathbb{C}[T_yX]}_y \rtimes \mathbb{C}\overline{W}_y \subset \widehat{H_{c,\eta}(T_yX, \overline{W}_y)}_0.$$

On the other hand, the diagonal terms agree with the the formal restrictions of the operators induced by

$$p_z(\delta u)^\vee \left((\delta v) + \frac{|\Gamma|}{2} \sum_{\gamma \in \Gamma_y \setminus \{1,z\}} \kappa_\gamma \frac{(\gamma \delta v + \delta v)^\vee}{\omega^\gamma} \gamma \right) p_z, \quad \delta \in T, \tag{3.6}$$

where Γ_y denotes the preimage of $\overline{W}_y$. The twisted differential operators induced by (3.6) are element of $H_{c,\eta}(X, \overline{W}_y)$, as we have seen from considering the case of cyclic groups above, so their restrictions to a formal neighborhood of y give elements of $\widehat{H_{c,\eta}(T_yX, \overline{W}_y)}_0$. Thus, we have checked that at every point, the formal completion of the image of $p_z\theta^{\mathrm{Dunkl}}p_z$ is contained in the formal completion of $\mathscr{H}_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$.

Finally, we use [5, Proposition 2.18] to obtain the global, twisting parameter:

$$\phi = \psi + \sum_{y \in \mathbb{P}^1 \setminus X'} \eta(y)\omega_y,$$

where ω_y is the class in $\mathbb{H}^2(\mathbb{P}^n, \Omega^{\geq 1}_{\mathbb{P}^n}) \simeq \mathbb{C}$ given by $\mathscr{O}(-y)$. Since y has degree 1, under the identification of §1.3 we have $\omega_y = -1$.

We have seen that the image of $p_z\theta^{\mathrm{Dunkl}}p_z$ inside $\Gamma(X', \mathscr{D}_{X'}^{\psi|_{X'}}) \rtimes \mathbb{C}\overline{W}$ is contained in the modified Cherednik algebra $H_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$. Thus the image is exactly $H_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$, since this is so at the level of the associated graded. Finally, we have explained that $H_{c,\phi}(\mathbb{P}^1, \overline{W}) \simeq H_{c,\eta,\psi}(\mathbb{P}^1, \overline{W})$, so the Proposition is proved. □

Remark 3.5. The key to the proof of Theorem 3.3 shows that operators of the form $p_z v^\vee D^u p_z$ induce Dunkl operators on $\mathbb{P}^1$. It is possible to replace these elements with members of a rank 2 Cherednik algebra corresponding to the subgroup $\widehat{\overline{W}}$ generated by complex reflections of the preimage of $\overline{W}$ under the map $GL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$. We may then use a result along the lines of [5, Example 2.20] to produce the Dunkl operators on $\mathbb{P}^1$.

Finally, we relate this construction to $H_c^{\mathfrak{so}(N)}(W)$. Recall that we began with $W \subset GL_3(\mathbb{C})$ and $c : \mathscr{S} \to \mathbb{C}$, W-equivariant. For $s \in \mathscr{S}$, we know that $c'|_Q(s, Y^+) = c'|_Q(s, Y^-) = c(s)$ for both points of Q, Y^+ and Y^-, fixed by s, while $c'|_Q$ is zero for other reflections. Furthermore, in the case when $-Id \in W$, under the isomorphism of (3.1), we have

$c''(\overline{s}, Y^+) = c''(\overline{s}, Y^-) = c(s)$, while c'' is zero for all other reflections of $\mathscr{S}(Q, W)$. Finally, recall from the beginning of this subsection that we have $\omega|_Q = 2\omega$.

By Theorem 3.3, we have $H_{c'',\omega|_Q}(Q, \overline{W}) = p_z A_\kappa(\Gamma) p_z$ (here we write $c'' = c'|_Q$ if $W = \overline{W}$), where

$$\kappa = |\Gamma|^{-1} + \sum_{\gamma \in \Gamma \setminus \{1\}} \kappa_\gamma \cdot \gamma,$$

with $\kappa_z = \frac{-(1+4\omega+4\sum_{s\in\mathscr{S}} c(s))}{|\Gamma|}$, $\kappa_\gamma = \frac{-2}{|\Gamma|} c(\overline{\gamma})$, if $\overline{\gamma} \in \mathscr{S}$, and $\kappa_\gamma = 0$ otherwise. This formula is simply obtained by inverting the equations (3.2) and (3.3).

Now Theorem 3.1 along with (3.1) allows us to relate the quotient of $H_c^{\mathfrak{so}(3)}(W)$ to the algebra $p_z A_\kappa(\Gamma) p_z$ with this parameter κ. We have left only to specify which $\Gamma \subset SL_2(\mathbb{C})$ corresponds to the given Coxeter group $W \subset GL_3(\mathbb{C})$. This is illustrated in the two tables below.

TABLE 1

W	$\overline{W}$	Γ
A_1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
I_m	D_m	$\mathbb{B}D_m$
$A_1 \times I_m$, m odd	D_{2m}	$\mathbb{B}D_{2m}$
A_3	$\mathbb{O}$	$\mathbb{BO}$

TABLE 2

W	$\overline{W}$	Γ
$A_1 \times I_m$, m even	D_m	$\mathbb{B}D_m$
B_3	$\mathbb{O}$	$\mathbb{BO}$
H_3	$\mathbb{D}$	$\mathbb{BD}$

The groups W are specified according to their Coxeter type. The other symbols are defined as follows: D_m refers to the dihedral group of order $2m$, $\mathbb{O} \simeq S_4$ is the tetrahedral group, and $\mathbb{D}$ refers to the dodecahedral group. The groups $\mathbb{B}D_m, \mathbb{BO}$, and $\mathbb{BD}$ are their double covers, and have types D_{m+2}, E_7, and E_8, respectively, under the well-known McKay correspondence.

Corollary 3.6.

(1) *If W is listed in Table 1, then for the corresponding Γ we have*

$$p_z A_\kappa(\Gamma) p_z \cong H_c^{\mathfrak{so}(3)}(W) / \left(H_\Omega + a\right),$$

where $a = (\omega - \sum_{s\in\mathscr{S}} c(s))(\omega + N - 2 - \sum_{s\in\mathscr{S}} c(s))$.

(2) *If W is listed in Table 2, then for the corresponding Γ we have that*

$$p_z A_\kappa(\Gamma) p_z \cong H_c^{\mathfrak{so}(3)}(W) / \left(H_\Omega + a, (-\mathrm{Id}) - 1\right).$$

Remark 3.7. In both cases of Corollary 3.6 we quotient out the central character of the algebra $H_c^{\mathfrak{so}(3)}(W)$. In general the centre of $H_c^{\mathfrak{so}(N)}(W)$ is generated by H_Ω and $\mathbb{C}$ if $(-\mathrm{Id}) \notin W$ (see [8]). If $(-\mathrm{Id}) \in W$ then it is easy to see that the centre is generated by $H_\Omega, \mathbb{C}$ and $(-\mathrm{Id})$. We note that in the latter case the generator $(-\mathrm{Id})$ is missing in [8, Theorem 9].

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