

Counting points on modular curves

Andrew V. Sutherland

Massachusetts Institute of Technology



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Why count points on modular curves?

- Curves with many points over finite fields
- Curves with no points over finite fields
- Hecke eigenvalues of modular forms
- Computing L -functions (special values, analytic ranks, ...)
- Sato-Tate distributions
- Identifying isogenies and morphisms (even non-modular ones)
- Computing endomorphism rings, decomposing modular Jacobians
- Mazur's "Program B"

Mazur's Bonn lectures and Program B

In the course of preparing my lectures for this conference, I found a proof of the following theorem, conjectured by Ogg (conjecture 1 [17b]):

THEOREM 1. Let ϕ be the torsion subgroup of the Mordell-Weil group of an elliptic curve E , over \mathbb{Q} . Then ϕ is isomorphic to one of the following 15 groups:

$$\begin{aligned} & \mathbb{Z}/m \cdot \mathbb{Z} && \text{for } m \leq 10 \text{ or } m = 12 \\ & \mathbb{Z}/2 \cdot \mathbb{Z} \times \mathbb{Z}/2\nu \cdot \mathbb{Z} && \text{for } \nu \leq 4 . \\ & && \vdots \end{aligned}$$

Theorem 1 also fits into a general program:

B. Given a number field K and a subgroup H of $GL_2 \hat{\mathbb{Z}} = \prod_p GL_2 \mathbb{Z}_p$ classify all elliptic curves E/K whose associated Galois representation on torsion points maps $\text{Gal}(\bar{K}/K)$ into $H \subset GL_2 \hat{\mathbb{Z}}$.

Galois representations attached to elliptic curves

Let E be an elliptic curve over a number field k . For each integer $N \geq 1$ the action of $G_k := \text{Gal}(\bar{k}/k)$ on $E[N]$ yields a **mod- N Galois representation**

$$\rho_{E,N}: G_k \rightarrow \text{Aut}(E[N]) \simeq \text{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Choosing a compatible system of bases and taking the inverse limit yields

$$\rho_E: G_k \rightarrow \text{GL}_2(\widehat{\mathbb{Z}}) \simeq \prod_{\ell} \text{GL}_2(\mathbb{Z}_{\ell}).$$

Theorem (Serre 1972)

For non-CM elliptic curves the image of ρ_E is an open subgroup $H_E \subseteq \text{GL}_2(\widehat{\mathbb{Z}})$.

There is thus a minimal positive integer M_E such that ρ_E factors through $\bar{\rho}_{E,M_E}$ and H_E is completely determined by its reduction modulo M_E .

There are infinitely many possibilities for M and H_E as E/k varies, but it is believed that only finitely many non-surjective projections to $\text{GL}_2(\mathbb{Z}_{\ell})$ arise, and only finitely many values of $[\text{GL}_2(\widehat{\mathbb{Z}}) : H_E]$ (even if only $[k : \mathbb{Q}]$ is fixed).

Modular curves

Let H be an open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}}) = \varprojlim \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) =: \varprojlim \mathrm{GL}_2(N)$.

Then H contains the kernel of $\pi_N: \mathrm{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(N)$ for some $N \geq 1$; the least such N is the **level** of H , and H is completely determined by $\pi_N(H)$.

Definition (Deligne-Rapoport 1973)

The **modular curves** X_H and Y_H are coarse spaces for the stacks \mathcal{M}_H and \mathcal{M}_H^0 that parameterize elliptic curves with **H -level structure**.

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- X_H is a smooth proper $\mathbb{Z}[\frac{1}{N}]$ -scheme with open subscheme Y_H .
The complement X_H^∞ of Y_H in X_H (the **cusps**) is finite étale over $\mathbb{Z}[\frac{1}{N}]$.
- When $\det(H) = \widehat{\mathbb{Z}}^\times$ the generic fiber of X_H is a **nice** curve X_H/\mathbb{Q} , and $X_H(\mathbb{C})$ is a Riemann surface isomorphic to $X_{\Gamma_H} := \Gamma_H \backslash \mathcal{H}$, where $\Gamma_H \subseteq \mathrm{SL}_2(\mathbb{Z})$ is the inverse image of $\pi_N(H) \cap \mathrm{SL}_2(N)$.
- In particular, $g(X_H) = g(X_{\Gamma_H})$, and X_H and X_{Γ_H} have the same cusps.
Note: $X_{\Gamma_H} = X_{\Gamma_{H'}} \not\Rightarrow X_H = X_{H'}$, and the levels of X_{Γ_H} and X_H may differ.
- If $\det(H) \neq \widehat{\mathbb{Z}}^\times$ then X_H is not geometrically connected (but that's OK!).

Classical modular curves

For $B_0(N) := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_2(N)$ we have $X_0(N) = X_{B_0(N)}$ (as curves over \mathbb{Q}).

For $B_1(N) := \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_2(N)$ we have $X_1(N) = X_{B_1(N)}$.

We similarly define $X_s(p)$, $X_{ns}(p)$, using Cartan subgroups $H \subseteq \mathrm{GL}_2(p)$.

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Example: Let us compute $\#X_1(13)(\mathbb{F}_{37})$.

Over \mathbb{F}_{37} there are 4 elliptic curves with a rational point of order 13:

$$\begin{aligned} y^2 &= x^3 + 4, & y^2 &= x^3 + 33x + 33, \\ y^2 &= x^3 + 8x, & y^2 &= x^3 + 24x + 22. \end{aligned}$$

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The genus 2 curve 169.1.169.1 is a smooth model for $X_1(13)$:

$$y^2 + (x^3 + x + 1)y = x^5 + x^4.$$

It has 23 rational points over \mathbb{F}_{37} .

What do these 23 points represent?

Moduli spaces of elliptic curves with H -level structure

Let H be an open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ of level N with image H in $\mathrm{GL}_2(N)$.
Let k be a perfect field whose characteristic does not divide N .

Definition

An H -level structure on an elliptic curve E/\bar{k} is the equivalence class $[\iota]_H$ of an isomorphism $\iota: E[N] \rightarrow (\mathbb{Z}/N\mathbb{Z})^2$, where $\iota \sim \iota'$ if $\iota = h \circ \iota'$ for some $h \in H$.

If we fix a basis so $E[N] := (\mathbb{Z}/N\mathbb{Z})^2$, then $[\iota]_H$ is a right H -coset in $\mathrm{GL}_2(N)$.

Definition

The set $Y_H(\bar{k})$ consists of equivalence classes of pairs $(E, [\iota]_H)$, where $(E, [\iota]_H) \sim (E', [\iota']_H)$ if there is an isomorphism $\phi: E \rightarrow E'$ for which the induced isomorphism $\phi_N: E[N] \rightarrow E'[N]$ satisfies $\iota \sim \iota' \circ \phi_N$.

Equivalently, $Y_H(\bar{k})$ consists of pairs $(j(E), \alpha)$, where $\alpha = HgA_E$ is a double coset in $H \backslash \mathrm{GL}_2(N) / A_E$, with $A_E := \{\varphi_N : \varphi \in \mathrm{Aut}(E)\}$.

The set of k -rational points $Y_H(k)$

The Galois group $G_k := \text{Gal}(\bar{k}/k)$ acts on $Y_H(\bar{k})$ by acting on coefficients of E and points in $E[N]$, which induces an action on $[\iota]_H$ and pairs $(E, [\iota]_H)$.

More precisely, $\sigma \in G_k$ send E to E^σ and induces an isomorphism $\sigma^{-1} : E^\sigma[N] \rightarrow E[N]$ defined by $(x : y : z) \mapsto (\sigma^{-1}(x) : \sigma^{-1}(y) : \sigma^{-1}(z))$.

For $P := (E, [\iota]_H) \in Y_H(\bar{k})$ we have $\sigma(P) := (E^\sigma, [\iota \circ \sigma^{-1}]_H)$.

The subset of G_k -stable points in $Y_H(\bar{k})$ forms the set of k -rational points $Y_H(k)$.

Lemma (DR73, Z15)

Each $P \in Y_H(k)$ is represented by a pair $(E, [\iota]_H) \in Y_H(k)$ with E defined over k , and any such a pair lies in $Y_H(k)$ if and only if for all $\sigma \in G_k$ there exists a $\varphi \in \text{Aut}(E_{\bar{k}})$ and an $h \in H$ such that

$$\iota \circ \sigma^{-1} = h \circ \iota \circ \varphi_N.$$

In other words, a pair $(j(E), \alpha)$ with $j(E) \in k$ and $\alpha = HgA_E$ lies in $Y_H(k)$ if and only if $Hg\sigma^{-1}A_E = HgA_E$ for all $\sigma \in G_k$, where $A_E := \{\varphi_N : \varphi \in \text{Aut}(E_{\bar{k}})\}$.

Interpreting rational points on Y_H

Recall that if E is an elliptic curve over a number field K , the action of G_K on torsion points of $E(\overline{K})$ yields a Galois representation

$$\rho_E: G_K \rightarrow \text{Aut}(E(\overline{K})_{\text{tor}}) \simeq \text{GL}_2(\widehat{\mathbb{Z}}) \simeq \varprojlim \text{GL}_2(N).$$

For each positive integer N , let $\rho_{E,N}$ denote the projection to $\text{GL}_2(N)$.

Lemma (DR73, RZB15)

Let H be an open subgroup of $\text{GL}_2(\widehat{\mathbb{Z}})$ of level N and let E be an elliptic curve over a number field K . There exists an isomorphism $\iota: E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$ such that $(E, [\iota]_H) \in Y_H(K)$ if and only if the image of $\rho_{E,N}$ is contained in a subgroup of $\text{GL}_2(N)$ conjugate to $\pi_N(H)$.

This is how we should understand the moduli interpretation of Y_H and X_H .

The set of k -rational cusps $X_H^\infty(k)$

Let $U(N) := \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -1 \rangle \subseteq \mathrm{GL}_2(N)$.

We define a right G_k -action on $H \backslash \mathrm{GL}_2(N)/U$ via $hgu \mapsto hg\chi_N(\sigma)u$, where $\chi_N(\sigma) := \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$ is defined by $\sigma(\zeta_N) = \zeta_N^e$.

Lemma (DR73)

The cardinality of $X_G^\infty(k)$ is equal to the cardinality of the subset of $H \backslash \mathrm{GL}_2(N)/U(N)$ fixed by $\chi_N(G_k)$.

When k is finite, we can compute both $\#X_H^\infty(k)$ and $\#Y_H(k)$ by counting the fixed points of a right G_k -action on a double coset spaces of $\mathrm{GL}_2(N)$.

We have

$$\#X_H(k) = \#(H \backslash \mathrm{GL}_2(N)/U(N))^{\chi_N(G_k)} + \sum_{j(E) \in k} \#(H \backslash \mathrm{GL}_2(N)/A_E)^{G_k}.$$

This does not depend on the choice of E or the choice of basis for $E[N]$.

Where the 23 points of $X_1(13)(\mathbb{F}_{37})$ come from

For $k = \mathbb{F}_{37}$ and the action of G_k is generated by the 37-power Frobenius σ , which induces the action of $\chi_{13}(G_k)$ on $\mu_{13}(\bar{k})$ and the Frobenius endomorphism π_E which acts on $E[13]$. We have

$$\# \mathrm{GL}_2(13) = 12^2 \cdot 13 \cdot 14, \quad \# B_1(13) = 12 \cdot 13, \quad \# U = 26,$$

and the right coset space $B_1(13) \backslash \mathrm{GL}_2(13)$ has cardinality $12 \cdot 14 = 168$.

- The space $B_1(13) \backslash \mathrm{GL}_2(13) / U(13)$ partitions $B_1(13) \backslash \mathrm{GL}_2(13)$ as $2^6 26^6$. These 12 double cosets correspond to the 12 cusps of $X_1(13)$.
The 6 partitions of size 26 are fixed by $\chi_{13}(\sigma) = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$ but not the others.
So we have **6 rational cusps**.
- The four elliptic curves E/\mathbb{F}_{37} with a points of order 13 have j -invariants 0, 16, 26, 35 (note $1728 \equiv 26 \pmod{37}$), so A_E is cyclic of order 6, 2, 4, 2.
The 168 right cosets of $B_1(13)$ correspond to the 168 points of order 13 in $E[13]$; in all 4 cases exactly 12 of these are fixed by π_E .
We thus get $2 + 6 + 3 + 6 =$ **17 non-cuspidal rational points**.

$$2 + 6 + 3 + 6 + 6 = 23$$

Counting \mathbb{F}_q -points on X_H

Let $j_H: X_H \rightarrow X(1)$ be the morphism induced by $H \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

$$\#X_G(\mathbb{F}_q) = \#X_H^\infty(\mathbb{F}_q) + \sum_{j \in \mathbb{F}_q} \#\{P \in Y_H(\mathbb{F}_q) : j_H(P) = j\}$$

Every term is computed by counting double cosets fixed by a right action. Computing $\chi_N(\sigma)$ is easy (reduce q modulo N). To compute π_E we use [DT02].

Theorem (DT02)

Let E/\mathbb{F}_q be an elliptic curve, and let π_E denote its Frobenius endomorphism. Define $a := \mathrm{tr} \pi_E = q + 1 - \#E(\mathbb{F}_q)$ and $R := \mathrm{End}(E) \cap \mathbb{Q}(\pi_E)$, let $\Delta := \mathrm{disc}(R)$ and $\delta := \Delta \bmod 4$, and let $b := \sqrt{(a^2 - 4q)/\Delta}$ if $\Delta \neq 1$ and $b := 0$ otherwise. The integer matrix

$$A_\pi := \begin{pmatrix} (a + b\delta)/2 & b \\ b(\Delta - \delta)/4 & (a - b\delta)/2 \end{pmatrix}$$

determines the action of π_E on $E[N]$ for all $N \geq 1$.

Note: A_π is determined only up to conjugacy, but we must compute A_E and A_π with respect to the same basis for $E[N]$.

Computational issues

- 1 Computing b typically requires determining $[\mathcal{O}_K : \text{End}(E)]$ where $K = \mathbb{Q}(\sqrt{a^2 - 4q})$. This is much harder than computing $\text{tr } \pi_E$.
The brute force approach tests $H_D(j(E)) \stackrel{?}{=} 0$ for discriminants D of all orders in \mathcal{O}_K containing $\mathbb{Z}[\pi_E]$. This is expensive and unnecessary.
We will enumerate every root of H_D for all such D as we enumerate $j(E)$!
- 2 Computing an explicit basis for $E[N]$ is painful when N is large; this only matters when $j(E) = 0, 1728$, but these two cases can get very expensive.

Solution to (1): Instead of enumerating j -invariants, enumerate Frobenius traces a and compute A_π for each triple (a, b, Δ) satisfying $4q = a^2 - b^2\Delta$. Then multiply the number of double cosets fixed by A_π by $h(D)$.

This reduces the problem to computing class numbers rather than Hilbert class polynomials, which is much easier (and can be done via table lookup).

Solution to (2): Instead of computing A_E , enumerate twists of elliptic curves with $j(E) = 0, 1728$ and compute A_π for each. No need to fix a basis for $E[N]$.

The algorithm

Given $H \subseteq \mathrm{GL}_2(N)$ and a prime power q , compute $X_H(\mathbb{F}_q)$ as follows:

- 1 Compute $f_H: \mathrm{GL}_2(N) \rightarrow \mathbb{Z}$ defined by $g \mapsto \#(H \backslash \mathrm{GL}_2(N) / \{\pm 1\})^g$.
Note that f_H does not depend on q and factors through the class map.
Indeed: $f_H(g) = [\mathrm{GL}_2(N) : H] \cdot \#(\pm H \cap g^{\mathrm{GL}_2(N)}) \cdot (\#g^{\mathrm{GL}_2(N)})^{-1}$.
- 2 Compute $n_\infty := \#X_H^\infty(\mathbb{F}_q) = \#(H \backslash \mathrm{GL}_2(N) / U(N))^{\chi_N(\sigma)}$.
(this step is fast in practice, but asymptotically annoying).
- 3 Compute $n_0 := \#j_H^{-1}(0)$ and $n_{1728} := \#j_H^{-1}(1728)$ by computing A_π for each twist, summing $f_H(A_\pi)$ values, and dividing by $\# \mathrm{Aut}(E_{\bar{k}})$.
- 4 Set $n_{\mathrm{ord}} := 0$ and for a from 1 to $\lfloor 2\sqrt{q} \rfloor$ coprime to q :
 - a Compute $D = a^2 - 4q$, put $D_0 := \mathrm{disc} \mathbb{Q}(\sqrt{D_\pi})$ and for $b^2 \mid (D/D_0)$:
 - 1 Set $D' := b^2 D_0$ and $\delta := D \bmod 4$ and compute A_π (for $D' < -4$).
 - 2 If $f_H(A_\pi) \neq 0$, compute/lookup $h(D)$ and add $f_H(A_\pi)h(D)$ to n_{ord} .
- 5 Compute n_{ss} by computing A_π for supersingular elliptic curves with $j \neq 0, 1728$ (only $a = 0, \pm 2q$ possible), and multiplying $f_H(A_\pi)$ by the counts of such curves (using $h(\sqrt{-q})$, $h(\sqrt{-4q})$ and [W69]).
- 6 Output $\#X_H(\mathbb{F}_q) = n_\infty + n_0 + n_{1728} + n_{\mathrm{ord}} + n_{\mathrm{ss}}$.

A non-trivial example

Consider the following genus 14 subgroup on the Mazur-B 5-adic list:

$$H := \left\langle \begin{pmatrix} 8 & 6 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 9 & 18 \\ 7 & 16 \end{pmatrix} \right\rangle \subseteq \mathrm{GL}_2(25);$$

this is the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(25)$. Counting points on X_H over \mathbb{F}_{2^r} , \mathbb{F}_{5^r} , \mathbb{F}_{7^r} for $1 \leq r \leq 14$ yields the L -polynomials

$$\begin{aligned} L_2(x) &= (2^2x^4 - 2x^3 + 3x^2 - x + 1)(2^2x^4 + 2x^3 + 3x^2 + x + 1)(2^8x^{16} + \cdots + 1), \\ L_3(x) &= (3x^2 - x + 1)^2(3x^2 + x + 1)^2(3^2x^4 + 9x^3 + 7x^2 + 3x + 1)(3^8x^{16} + \cdots + 1), \\ L_7(x) &= (7^2x^4 - 7x^3 + 13x^2 - x + 1)(7^2x^4 + 7x^3 + 13x^2 + x + 1)(7^2x^4 + 7x^3 + 3x^2 + x + 1) \\ &\quad (7^8x^{16} + \cdots + 1), \end{aligned}$$

suggesting the \mathbb{Q} -isogeny decomposition of the Jacobian has shape 2-2-2-8. Hashing traces and searching for 5-power conductor genus 2 curves yields

$$\begin{aligned} y^2 + (x^3 + x + 1)y &= -3x^4 + 7x^3 + x^2 - 5x + 1, \\ y^2 + (x^3 + x + 1)y &= x^6 - 13x^4 + 37x^3 + 6x^2 - 23x + 6, \\ y^2 + (x^3 + x + 1)y &= 6x^6 - 5x^5 + 12x^4 - 13x^3 + 6x^2 - 13x - 4, \end{aligned}$$

each of which have RM by $\mathbb{Q}(\sqrt{5})$ and Jacobians of Mordell-Weil rank 2.

Complexity analysis

We can use sub-exponential time Monte-Carlo algorithms to compute class numbers and still get a provably correct result (in practice we just look up class numbers in a precomputed table).

As written, the complexity of this algorithm is

$$N^{4+o(1)} + q^{1/2+o(1)}N^{o(1)}.$$

The constant factors are very small (the inner loop is just table lookups). The dependency on N can easily be improved to $N^{3+o(1)}$, and even to $N^{2+o(1)}$ for suitable H (this is work in progress).

If we wish to compute $\#X_H(\mathbb{F}_q)$ for many values of q (for example, all primes $p \nmid N$ up to some bound B), the computation of $f_H: \mathrm{GL}_2(N) \rightarrow \mathbb{Z}$ only needs to be done once, and we can precompute all the class numbers up to $4B$ in $O(B^{3/2+o(1)})$ time (deterministically) by counting binary quadratic forms.

Corollary: We can completely determine the L -function $L(X_H, s)$ in time $\mathrm{cond}(\mathrm{Jac}(X_H))^{3/4+o(1)}$, which is $N^{3g/4+o(1)}$ or $N^{3g/2+o(1)}$ (depending on H).

Performance comparison

B	Pari/GP v2.11					new algorithm				
	$N=11$	47	97	149	197	$N=11$	47	97	149	197
2^{12}	0.1	0.1	0.2	0.2	0.3	0.1	0.1	0.1	0.1	0.1
2^{13}	0.3	0.3	0.3	0.3	0.4	0.1	0.1	0.1	0.1	0.1
2^{14}	0.7	0.7	0.8	0.8	0.9	0.1	0.1	0.2	0.2	0.2
2^{15}	2.0	2.0	2.1	2.1	2.2	0.2	0.3	0.3	0.5	0.4
2^{16}	6.1	5.9	6.2	5.9	6.1	0.5	0.6	0.9	1.2	1.1
2^{17}	18	17	18	18	18	1.3	1.8	5.3	3.1	3.1
2^{18}	54	51	54	51	51.8	3.8	5.1	9.9	9.0	8.0
2^{19}	159	153	160	153	453	14	15	21	25	21
2^{20}	462	455	462	450	453	36	42	53	62	57
2^{21}	1350	1340	1350	1330	1340	107	120	146	160	156
2^{22}	?	?	?	?	?	333	353	399	427	433
2^{23}	?	?	?	?	?	950	1020	1120	1180	1220
2^{24}	?	?	?	?	?	2840	2990	3210	3360	3500

Intel Skylake 2.0 GHz CPU times (seconds)

Zeta functions and L -functions

Let X/\mathbb{Q} be a **nice** (smooth, projective, geometrically integral) curve of genus g . For primes p of good reduction (for X) we have a **zeta function**

$$Z(X_p; s) := \exp\left(\sum_{r \geq 1} \#X_p(\mathbb{F}_{p^r}) \frac{T^r}{r}\right) = \frac{L_p(T)}{(1-T)(1-pT)},$$

in which the **L -polynomial** $L_p \in \mathbb{Z}[T]$ in the numerator satisfies

$$L_p(T) = T^{2g} \chi_p(1/T) = 1 - a_p T + \cdots + p^g T^{2g};$$

here $\chi_p(T)$ is the charpoly of the Frobenius endomorphism of $\text{Jac}(X_p)$ (this implies $\#\text{Jac}(X_p) = L_p(1)$, for example). The **L -function** of X is

$$L(X, s) = L(\text{Jac}(X), s) := \sum_{n \geq 1} a_n n^{-s} := \prod_p L_p(p^{-s})^{-1},$$

where the Dirichlet coefficients $a_n \in \mathbb{Z}$ are determined by the $L_p(T)$. In particular, $a_p = p + 1 - \#X_p(\mathbb{F}_p)$ is the **trace of Frobenius**.

The Selberg class with polynomial Euler factors

The **Selberg class** S^{poly} contains Dirichlet series $L(s) = \sum_{n \geq 1} a_n n^{-s}$ satisfying:

- 1 $L(s)$ has an **analytic continuation** that is holomorphic at $s \neq 1$;
- 2 For some $\gamma(s) = Q^s \prod_{i=1}^r \Gamma(\lambda_i s + \mu_i)$ and ε , the completed L -function $\Lambda(s) := \gamma(s)L(s)$ satisfies the **functional equation**

$$\Lambda(s) = \varepsilon \overline{\Lambda(1 - \bar{s})},$$

where $Q > 0$, $\lambda_i > 0$, $\text{Re}(\mu_i) \geq 0$, $|\varepsilon| = 1$. Define $\deg L := 2 \sum_i \lambda_i$.

- 3 $a_1 = 1$ and $a_n = O(n^\epsilon)$ for all $\epsilon > 0$ (**Ramanujan conjecture**).
- 4 $L(s)$ has an **Euler product** $L(s) = \prod_p L_p(p^{-s})^{-1}$ in which each local factor $L_p \in \mathbb{Z}[T]$ has degree at most $\deg L$.

For any nice curve X the Dirichlet series $L_{\text{an}}(s, X) := L(X, s + \frac{1}{2})$ satisfies both (3) and (4) (by Weil), and conjecturally lies in S^{poly} .

For modular curves we also know (1) and (2), so $L(s, X_H) \in S^{\text{poly}}$.

Strong multiplicity one

Theorem (Kaczorowski-Perelli 2001)

If $A(s) = \sum_{n \geq 1} a_n n^{-s}$ and $B(s) = \sum_{n \geq 1} b_n n^{-s}$ lie in S^{poly} and $a_p = b_p$ for all but finitely many primes p , then $A(s) = B(s)$.

Corollary

If $L_{\text{an}}(s, X)$ lies in S^{poly} then it is determined by (any choice of) all but finitely many coefficients a_p . In particular, all of the local factors are completely determined by the Frobenius traces a_p at good primes.

Henceforth we assume that $L_{\text{an}}(s, X) \in S^{\text{poly}}$.

Let $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^s \Gamma(s)$, and define $\Lambda(X, s) := \Gamma_{\mathbb{C}}(s)^g L(X, s)$. Then

$$\Lambda(X, s) = \varepsilon N^{1-s} \Lambda(X, 2-s).$$

where the **analytic root number** $\varepsilon = \pm 1$ and **analytic conductor** $N \in \mathbb{Z}_{\geq 1}$ are also determined by the Frobenius traces a_p at good primes.

Effective strong multiplicity one

Fix a finite set of primes \mathcal{S} (e.g. bad primes) and an integer M that we know is a multiple of the conductor N (e.g. $M = \Delta(X)$).

There is a finite set of possibilities for $\varepsilon = \pm 1$, $N|M$, and the Euler factors $L_p \in \mathbb{Z}[T]$ for $p \in \mathcal{S}$ (the coefficients of $L_p(T)$ are bounded).

Suppose we know the a_n for all $n \leq c_1 \sqrt{M}$ with $p \nmid n$ for $p \in \mathcal{S}$. For a suitably large c_1 , exactly one choice of ε , N , and $L_p(T)$ for $p \in \mathcal{S}$ will make it possible for $L(X, s)$ to satisfy its functional equation.

One can explicitly determine a set of $O(N^\epsilon)$ candidate values of c_1 , one of which is guaranteed to work; in practice the first one usually works.

This gives an effective algorithm to compute ε , N , and $L_p(T)$ for $p \in \mathcal{S}$, provided we can compute $L_p(T)$ at good $p \leq B$, where $B = O(\sqrt{N})$.

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