

### 3. Hochschild-Serre spectral sequence for $SL(3, \mathbb{R})$

The first big chart here is the  $E_1$  term of the Hochschild-Serre spectral sequence for computing the  $\mathfrak{n}$ -cohomology of a Harish-Chandra module  $X$  for  $SL(3, \mathbb{R})$ . Each column shows the contribution of one representation of  $K = SO(3)$ ; the numbers are the weights of the representation of  $T_1 = SO(2)$ . For example, the column “5-diml” says that each copy of the 5-dimensional representation of  $SO(3)$  occurring in  $X$  contributes a three-dimensional piece to the degree one part of  $E_1$ ; the corresponding weights of  $T_1$  are 1, 0, and  $-3$ .

degree	1-diml		3-diml		5-diml		7-diml		9-diml		11-diml	
0	0		1		2		3		4		5	
1	-1, -2	-1	0, -1	-2	1, 0	-3	2, 1	-4	3, 2	-5	4, 3	-6
2	-3	-2, -3	-2	-3, -4	-1	-4, -5	0	-5, -6	1	-6, -7	2	-7, -8
3		-4		-5		-6		-7		-8		-9

Suppose we concentrate for a moment on the  $-3$  weight space (which has one dimension from each copy of the 5-dimensional representation of  $K$ ). This weight space might fail to survive to  $E_\infty$  in two ways. First, it could be in the image of a differential from degree 0. But we see from the table (extrapolated infinitely to the right) that the weight  $-3$  does not appear in degree 0; so this is impossible. Second, our weight space could have a non-zero image in the degree 2. By inspection of the degree 2 row of the table, we see that the weight  $-3$  appears there only if  $X$  has a copy of the  $SO(3)$  representation of dimension 1 or the  $SO(3)$  representation of dimension 3.

So inspection of this spectral sequence leads to the following conclusion.

**Proposition 1** *Suppose  $X$  is a Harish-Chandra module for  $SL(3, \mathbb{R})$ , and that the  $SO(3)$  representations of dimensions 1 and 3 do not occur in  $X$ . Then the  $T_1$  weight  $-3$  appears in  $H^\bullet(\mathfrak{n}, X)$  only in degree 1, where it has multiplicity equal to the multiplicity of the 5-dimensional representation of  $SO(3)$  in  $X$ .*

An identical argument proves

**Proposition 2** *Suppose that  $2m + 1 \geq 7$ , and that the  $SO(3)$  representations of dimensions  $2m - 1$ ,  $2m - 3$ , and  $2m - 5$  do not occur in  $X$ . Then the  $T_1$  weight  $-m - 1$  appears in  $H^\bullet(\mathfrak{n}, X)$  only in degree 1, where it has multiplicity equal to the multiplicity of the  $2m + 1$ -dimensional representation of  $SO(3)$  in  $X$ .*

How was this spectral sequence constructed? Recall that the  $E_1$  term in general is built from pieces  $\bigwedge^p(\mathfrak{n} \cap \mathfrak{s})^* \otimes H^q(\mathfrak{n}_\mathfrak{t}, X)$ , and that the degree is  $p + q$ . The top entries in the two columns for each representation of  $K$  are the weights in  $H^0(\mathfrak{n}_\mathfrak{t}, \bullet)$  and  $H^1(\mathfrak{n}_\mathfrak{t}, \bullet)$  for the indicated representation of  $K$ . (Those are provided by the Bott-Kostant theorem.) The rest of each column comes from adding the weights of  $\bigwedge^\bullet(\mathfrak{n} \cap \mathfrak{s})^*$ , which are tabulated below.

degree	$\bigwedge^\bullet(\mathfrak{n} \cap \mathfrak{s})^*$
0	0
1	-1, -2
2	-3