18.704: Seminar in Algebra and Number Theory
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General parabolic subgroups of $Sp(2n)$

Let V be a $2n$-dimensional vector space over a field F, B be a non-degenerate symplectic form of V, $Sp(2n)$ be the set $\{g \in GL(V) \mid B(gv, gw) = B(v, w) \quad (v, w \in V)\}$, S_1 be an arbitrary isotropic subspace of V, and $\dim(S_1) = k_1$.

For $r \in \mathbb{Z}^+$, we arbitrarily pick subspaces $S_2, S_3, \ldots, S_r$ such that

$$S_r \subset S_{r-1} \subset \ \ldots \ \subset S_2 \subset S_1$$

Since, S_1 is isotropic, it follows that each S_i $(2 \leq i \leq r)$ is also isotropic. We see that this is ture because since S_1 is isotropic, $B(v, w) = 0$ for all $v, w \in S_1$. And, since each S_i $(2 \leq i \leq r)$ is a subspace of S_1 then $B(v', w') = 0$ for all $v', w' \in S_i$, thereby making S_i isotropic.

Let $\dim(S_i) = k_i$ $(2 \leq i \leq r)$. Then, we know that

$$k_r \leq k_{r-1} \leq \ \ldots \ \leq k_2 \leq k_1$$

We now define $S_i^\perp$ to be the orthogonal complement of S_i $(1 \leq i \leq r)$. Therefore, it is clear that

$$S_r \subset S_{r-1} \subset \ \ldots \ \subset S_2 \subset S_1 \subset S_1^\perp \subset S_2^\perp \subset \ \ldots \ \subset S_r^\perp$$

Let $W = S_1^\perp / S_1$ to be a set of cosets of the form $v + S_1$ where $v \in S_1^\perp$. We know that $\dim(W) = \dim(S_1^\perp/S_1) = \dim(S_1^\perp) - \dim(S_1)$. Since $\dim(S_1) = k_1$, we also know that $\dim(S_1^\perp) = \dim(V) - k_1 = 2n - k_1$. Let $k = k_1$. Therefore

$$\begin{aligned}
\dim(W) &= \dim(S_1^\perp/S_1) \\
&= \dim(S_1^\perp) - \dim(S_1) \\
&= 2n - k - k \\
&= 2(n - k)
\end{aligned}$$

A stabilizer P for $S_r \subset S_{r-1} \subset \ \dots \ \subset S_2 \subset S_1$ is defined as

$$P(S_r \subset \ \dots \ \subset S_1) = \{g \in Sp(V) \ | \ gS_r = S_r, \dots, gS_1 = S_1\}$$

To give a homomorphism f from a group P to a direct product group $G_r \times G_{r-1} \times \ \dots \ \times G_1 \times G_0$ is the same as giving $r+1$ separate homomorphisms $f_i : P \to G_i$ where $i = 0, 1, \dots, r$. The relationship is:

$$f(p) = (f_r(p), f_{r-1}(p), \dots, f_0(p)) \ \to G_r \times \ \dots \ \times G_0$$

We now define the following group homomorphism:

$$f : P(S_r \subset \ \dots \ \subset S_1) \to GL(S_r) \times GL(S_{r-1}/S_r) \times \ \dots \ \times GL(S_1/S_2) \times Sp(W)$$

as a coordinate homomorphism:

$$\begin{aligned} &\text{If } g \in P, \text{ then} \\ f_r(g) &= g|_{S_r} \in GL(S_r) \\ f_{r-1}(g) &= g|_{S_{r-1}/S_r} \in GL(S_{r-1}/S_r) \\ &\vdots \\ f_1(g) &= g|_{S_1/S_2} \in GL(S_1/S_2) \\ f_0(g) &= g|_{S_1^{\perp}/S_1} \in Sp(S_1^{\perp}/S_1) \end{aligned}$$

Note that we use the notation $g|_A$ to mean the restriction of g to the subspace A.

Let N_f be the kernel of f. We know that the kernel of a group homomorphism is a normal subgroup. Thus, N_f is a normal subgroup of $P(S_r \subset \ \dots \ \subset S_1)$.

Let N be a normal subgroup of $S_r \subset \ \dots \ \subset S_1$ (denoted $N(S_r \subset \ \dots \ \subset S_1)$) which is defined as:

$$\begin{aligned} &g \in Sp(V) \text{ such that} \\ g(x_r) &= x_r \quad (x_r \in S_r), \\ g(x_{r-1}) &= x_{r-1} + (\text{an element of } S_r) \quad (x_{r-1} \in S_{r-1}), \\ &\vdots \\ g(x_1) &= x_1 + (\text{an element of } S_2) \quad (x_1 \in S_1) \\ g(w) &= w + (\text{an element of } S_1) \ (w \in S_1^{\perp}) \end{aligned}$$

With the help of the class, I outlined some examples in lecture showing that the above definition defines a normal subgroup of $P(S_r \subset \ \dots \ \subset S_1)$. Furthermore, from the definition of f and N_f above, we see that $N_f = N(S_r \subset \ \dots \ \subset S_1)$.

It then follows that f defines an inclusion:

$$P(S_r \subset \ \ldots \ \subset S_1)/N(S_r \subset \ \ldots \ \subset S_1) \longrightarrow GL(S_r) \times GL(S_{r-1}/S_r) \times \ \ldots \ \times GL(S_1/S_2) \times Sp(W)$$

which is also surjective. Note: the proof of surjectivity is left out of these notes; it involves finding a number of elements of P and showing that every element of $GL(S_r) \times GL(S_{r-1}/S_r) \times \ \ldots \ \times GL(S_1/S_2) \times Sp(W)$ is an image (under f) of elements of P/N.

Therefore, f is a bijective map from $P(S_r \subset \ \ldots \ \subset S_1)/N(S_r \subset \ \ldots \ \subset S_1)$ to $GL(S_r) \times GL(S_{r-1}/S_r) \times \ \ldots \ \times GL(S_1/S_2) \times Sp(W)$, which implies that

$$P(S_r \subset \ \ldots \ \subset S_1)/N(S_r \subset \ \ldots \ \subset S_1) \cong GL(S_r) \times GL(S_{r-1}/S_r) \times \ \ldots \ \times GL(S_1/S_2) \times Sp(W)$$

Note the difference between this result and the one demonstrated for maximal parabolic subgroups of $Sp(2n)$ in a previous talk. Specifically, for maximal parabolic subgroups:

$$P(S)/N = GL(k,F) \times S(W)$$

whereas for general parabolic subgroups, we have

$$P(S_r \subset \ \ldots \ \subset S_1)/N(S_r \subset \ \ldots \ \subset S_1) \cong GL(S_r) \times GL(S_{r-1}/S_r) \times \ \ldots \ \times GL(S_1/S_2) \times Sp(W)$$