18.757 HW QUASI-ANSWERS

1. 1st Homework

Problem 1. Prove that the restriction of homogeneous polynomials to sphere is one to one. That is, if Q is a homogeneous polynomial of degree m and $Q|_{S^{n-1}} = 0$, then $Q = 0$.

Proof. This is true only if the sphere is nonempty; that is, if $n > 0$. So assume that. For any nonzero vector $v \in \mathbb{R}^n$, $\frac{v}{\|v\|} \in S^{n-1}$. If Q is a homogeneous polynomial of degree m, and $Q|_{S^{n-1}} = 0$, then $Q(v) = Q(\|v\|\frac{v}{\|v\|}) = \|v\|^m Q(\frac{v}{\|v\|}) = 0$. So Q vanishes except perhaps at 0. Since Q is continuous, we have $Q = 0$.

□

Problem 2. Prove that $P^{m-2} \subset P^m$ for $m \geq 2$.

Proof. Recall that P^m is the space of restrictions to the sphere of homogeneous polynomials of degree m. Because of Problem 1, it makes sense to define $\mathsf{i} : P^{m-2} \to P^m$ by $\mathsf{i}(f|S^{n-1}) = [(x_1^2 + \cdots + x_n^2)f]|S^{n-1}$. It is clear that $(x_1^2 + \cdots + x_n^2)f|_{S^{n-1}} \in P^m$. Since multiplication by a nonzero polynomial is one to one, this map is one to one. Hence $P^{m-2} \subset P^m$.

□

Problem 3. Calculate $\dim P_m$ and $\dim P_m/P_{m-2}$

Proof. By Problem 1, the monomials of degree m in n variables form a basis of P_m. Hence $\dim P_m = \binom{n+m-1}{m} = \binom{n+m-1}{n-1}$, and $\dim P_m/P_{m-2} = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$.

□

2. 2nd Homework

Problem 4. Calculate the eigenspaces of $r^2\Delta$ acting on $S^m(\mathbb{R}^n)$

Proof. For $n = 1$, x^k are the eigenvectors with eigenvalue $k(k-1)$. Now we consider $n \geq 2$. We can show from a brute calculation that

$$r^2\Delta(r^{2k}f) = 2k(n + 2m - 2k - 2)r^{2k}f + r^{2k+2}\Delta(f).$$

Therefore $r^{2k}H^{m-2k}$ is an eigenspace of $r^2\Delta$ with eigenvalue $2k(n+2m-2k-2)$. Observe that we have a decomposition $S^m(\mathbb{R}^n) = H^m \oplus r^2H^{m-2} \oplus \cdots \oplus r^{2[\frac{m}{2}]}H^{m-2[\frac{m}{2}]}$, hence this is an eigenspace decomposition of $S^m(\mathbb{R}^n)$. Furthermore the eigenvalues $2k(n+2m-2k-2)$ are all distinct for $n \geq 2$. Therefore $\{r^{2k}H^{m-2k}|k = 0\ldots[\frac{m}{2}]\}$ is the complete set of eigenspaces of $r^2\Delta$ on $S^m(\mathbb{R}^n)$.

□

Problem 5. Show that every n variable polynomial can be uniquely written as a polynomial in r^2 with harmonic polynomial coefficients.

Proof. First, observe that any polynomial f can be uniquely written as a sum of homogeneous polynomials $f = f_0 + \ldots f_d$, $f_i \in S^i(\mathbb{R}^n)$. Second, we have the decomposition $S^i(\mathbb{R}^n) = H^i \oplus r^2H^{i-2} \oplus \cdots \oplus r^{2[\frac{i}{2}]}H^{i-2[\frac{i}{2}]}$ for each i. The assertion follows.

□

Problem 6. Suppose $n = p + q$, positive integers. Decompose ${}^nH^m|_{O(p)\times O(q)}$ into irreducibles.

Proof. We have the decomposition $S^m(\mathbb{R}^n)|_{O(p)\times O(q)} = \bigoplus_{i=1}^m S^i(\mathbb{R}^p)\otimes S^{m-i}(\mathbb{R}^q)$. Express both sides as a sum of ${}^iH^j$'s using $S^i(\mathbb{R}^n) = H^i \oplus r^2H^{i-2}\oplus\cdots\oplus r^{2[\frac{i}{2}]}H^{i-2[\frac{i}{2}]}$. Since ${}^pH^i$ and ${}^qH^j$ are irreducible representations of $O(p)$ and $O(q)$ respectively, ${}^pH^i \otimes^q H^j$ is an irreducible representation of $O(p)\times O(q)$. Hence, after cancellation, we have

$$
{}^nH^m|_{O(p)\times O(q)} = \bigoplus_{\substack{0\le i,j\le m\\ i+j\equiv m \pmod 2}} {}^pH^i \otimes {}^qH^j
$$

□

3. 3rd Homework

Problem 7. Suppose $\{\lambda_k \in \mathbb{C}|k \in \mathbb{N}\}$ is a sequence of complex numbers. We may try to define a map $T : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by the formula $(a_k) \mapsto (\lambda_k a_k)$. Then,

(a) T is well-defined if and only if (λ_k) is bounded.
(b) T is compact if and only if $\lambda_k \to 0$.

Proof. (a) If (λ_k) is not bounded, there is a subsequence (λ_{k_i}) s.t. $i \le |\lambda_{k_i}|$. Let b be the vector with $\frac{1}{i}$ at each k_ith coordinate and 0 at all other places. Clearly b is in $l^2(\mathbb{C})$, but $T(b)$ is not.

Conversely, if $\lambda_k \le M$ for all k, then $\|T(a)\| \le M\|a\|$, hence $T(a) \in l^2(\mathbb{C})$.

(b) If $\lambda_k \nrightarrow 0$, then there is an $\epsilon > 0$ and a subsequence (λ_{k_i}) s.t. $|\lambda_{k_i}| > \epsilon$ for some ϵ. Let b_i be the vector with 1 at the m_ith coordinate and 0 elsewhere. Then $\{b_i\}$'s are all in the unit ball, but the sequence of vectors $\{T(b_i)\}$ has no convergent subsequence. Hence T is not compact. Conversely, suppose $\lambda \to 0$. Define $T_i : l^\infty(\mathbb{C}) \to l^\infty(\mathbb{C})$ by $\sum a_k e_k \mapsto \sum_{k\le i} \lambda_k a_k e_k$. Then T_i's are compact operators. Also, $||T - T_i|| = sup_{k>i}|\lambda_k|$, so $T_i \to T$ in the operator norm topology. Hence T is compact.

□

Problem 8. let $G \subset \mathbb{H}^\times$ be the Lie group of unit quaternions, which is diffeomorphic to S^3. The quaternionic product defines left and right actions of G on $\mathbb{H} \simeq \mathbb{R}^4$, defining a map $\Phi : G \times G \to O(4)$. Recall that $L^2(S^3) = L^2(G) = \bigoplus_{m\ge 0} {}^4H^m$ as $O(4)$-modules, hence as $(G \times G)$-modules. Moreover, Peter-Weyl tells us that $L^2(G) = \bigoplus_{\pi\in\hat{G}} V_\pi \otimes V_\pi^*$ as $G \times G$-modules. Reconcile these two orthonormal decompositions.

Proof. The given two actions of $G\times G$ on $L^2(S^3)$ coincide. It can be shown that $\Phi^*({}^4H^m)$'s are distinct irreducible $G \times G$ representations, so by Schur's lemma, the given two decompositions are the same (up to some permutation). More precisely, observe that Φ is onto the identity component of $O(4)$ which is $SO(4)$(one can compute its kernel $\{(1, 1), (-1, -1)\}$). Also ${}^4H^m$ is irreducible as $SO(4)$ representation, so $\Phi^*({}^4H^m)$ is irreducible as $G \times G$ representation.

[Additional hint for proving that ${}^4H^m$ remains irreducible on restriction to $SO(4)$: we know that ${}^4H^m$ restricted to $SO(3)$ is the sum of the representations ${}^3H^k$ restricted to $SO(3)$, for $0 \le k \le m$. Because $SO(3)$ acts transitively on S^2, the trivial representation of $SO(3)$ can appear in functions on S^2 only in the constant functions. That is, ${}^3H^k$ contains the trivial representation of $SO(3)$ only if $k = 0$. It follows that ${}^4H^m|_{SO(3)}$ contains at most one copy of the trivial representation of $SO(3)$. Now copy the proof given in class that ${}^4H^m$ is irreducible for $O(4)$.]

We can reconcile these two decompositions more directly. Since $\Phi^*({}^4H^m)$ is an irreducible $G\times G$ representation, $\Phi^*({}^4H^m) = V\otimes W$ for some irreducible G representations V and W. Let $\sigma : G\times G \to G\times G$ be the interchanging map of two factors, and $\tau : O(4) \to O(4)$ be the inner automorphism defined by $\tau(r)x = \overline{r(\overline{x})}$ for $r \in O(4)$, $x \in \mathbb{H}$, and $\overline{x}$ the conjugation of x in $\mathbb{H}$. Then we have $\Phi\circ\sigma = \tau\circ\Phi$, hence $\sigma^*\Phi^*({}^4H^m) = \Phi^*\tau^*({}^4H^m) \cong \Phi^*({}^4H^m)$ as τ is an inner automorphism. Therefore $V\otimes W \cong \Phi^*({}^4H^m) \cong \sigma^*\Phi^*({}^4H^m) \cong W\otimes V$ as $G\times G$ representations. Moreover, ${}^4H^m$ is self dual as $O(4)$ representation, so $V\otimes W \cong \Phi^*({}^4H^m) \cong \Phi^*({}^4H^{m*}) \cong \Phi^*({}^4H^m)^* \cong (V\otimes W)^* \cong V^*\otimes W^*$. So we have $V \cong V^*$, $W \cong W^*$. As a result, we finally get $\Phi^*({}^4H^m) \cong V\otimes V^*$ for some irreducible G representation V. ☐

Problem 9. Find a two-dimensional (complex) representation of a Lie group that is not a direct sum of irreducible representations.

Proof. Let $G = \mathbb{R}$ act on $V = \mathbb{C}^2$ by $\pi(r)(x,y) = (x+ry, y)$. Equivalently, G is the subgroup of $GL(2,\mathbb{C})$ consist of the matrices of the form $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. Then, the only proper G-invariant subspace is the one spanned by (1,0). Hence V is neither irreducible nor direct sum of irreducible representations.

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4. 4th Homework

Problem 10. Suppose G is a compact group. Prove that $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthonormal basis for $L^2_{\text{class}}(G)$.

Proof. From Schur orthogonality, we know that $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthonormal set. From Peter-Weyl, we know that they span a dense subset of $L^2_{\text{class}}(G)$. Hence the assertion follows. ☐

Problem 11. Suppose that G is a finite group. Then,

(a) # conjugacy classes of G = # irreducible representations;
(b) $\Sigma_{\pi\in\hat{G}}(\dim\pi)^2 = |G|$;
(c) # 1-dimensional representations of $G = |G/[G,G]|$.

Proof. (a) Since G is finite, $L^2_{\text{class}}(G)$ is the space of all class functions, hence the dimension of it is just the number of conjugacy classes of G. On the other hand, by the previous problem, its dimension is the number of irreducible representations of G.

(b) Since G is finite, $L^2(G)$ is the same as the space of all complex functions on G, hence the dimension of it is just $|G|$. On the other hand, by Peter-Weyl, we have $L^2(G) = \bigoplus_{\pi\in\hat{G}} V_\pi\otimes V_\pi^*$. Hence the dimension is $\Sigma_{\pi\in\hat{G}}(\dim\pi)^2$

(c) Giving a 1-dimensional representation is equivalent to giving a group homomorphism $G \to GL(\mathbb{C})$. Since $GL(\mathbb{C})$ is abelian, this is equivalent to giving a homomorphism $G/[G,G] \to GL(\mathbb{C})$, which is again equivalent to giving a 1-dimensional representation of $G/[G,G]$. All irreducible representations of $G/[G,G]$ are 1-dimensional, so the assertion follows.

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Problem 12. List all irreducible representations of $G = S_3$, and $G = \{\pm1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$.

Proof. 1) $G = S_3$: The conjugacy classes of G is characterized by permutation type, hence they are $\{id\}, \{(12),(13),(23)\}, \{(123),(132)\}$. And $G/[G,G] = \{\pm1\}$. By the previous problem, there are 3 irreducible representations of G, and 2 of them are 1-dimensional. We have trivial representation and sign representation which are nonequivalent 1-dimensional representations. Consider the defining representation of $G = S_3$ on $\mathbb{C}^3 = \mathbb{C}\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. We have 1 copy of the trivial representation $\mathbb{C}\{\mathbf{1} + \mathbf{2} + \mathbf{3}\}$ in it. Define a G-invariant inner product: $< \mathbf{i}, \mathbf{j} >= \delta_{i,j}$. Now the complement of the trivial representation if also a G-submodule. While S_3 is not abelian, the defining representation is faithful, so it cannot be decomposed into 1-dimensional representations. Therefore, the complement of the trivial representation is the 2-dimensional irreducible representation.

2) $G = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$: Similarly to the above problem, we find the conjugacy classes of G, which are $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$. And $|G/[G, G]| = 4$. Hence there are 5 irreducible representations and 4 of them are 1-dimensional. The 1-dimensional representations are given by the trivial representation and $\pi(\pm 1) = -\pi(\pm i) = -\pi(\pm j) = \pi(\pm k) = 1$ and similarly 2 more. We can view quaternion $\mathbb{H}$ as $\{a + jb | a, b \in \mathbb{C}\}$, so $\mathbb{C}$ vector space (by right multiplication). This naturally gives a faithful 2-dimensional G representation (by left multiplication), which is irreducible because of the similar reasoning as in 1). $\square$