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1 Introduction

Quite a bit of the serious mathematical theory of self-adjoint operators was created to serve the needs of quantum mechanics. These notes are a quick-and-dirty outline of the simplest mathematical setting of quantum mechanics. None of it should be taken too seriously: real physics is hard, and requires more than a few nice mathematical ideas.

2 Classical physics

In order to describe a toy picture of quantum mechanics, I’ll start with a toy picture of classical mechanics. In the toy picture, the state of a physical system is described by a point $x$ belonging to a state space $X$. For example, if we want to study the earth going around the sun classically, we might think of the universe as $\mathbb{R}^3$, and the sun as fixed at the center of the universe (and therefore at the point $0 \in \mathbb{R}^3$). The state of the earth is then given by two vectors in $\mathbb{R}^3$: its position $p$ and its velocity $v$. The state of the earth is the pair of vectors

$$x = (p, v) \in \mathbb{R}^3 \oplus \mathbb{R}^3 \simeq \mathbb{R}^6 = X. \quad (2.1)$$
Everything we want to know about the earth (assuming we take a sufficiently superior attitude toward television and so on) is recorded in the pair of vectors \((p, v) \in \mathbb{R}^6\).

Of course the state of the earth changes with time: it moves, with a changing velocity. What we are really interested in is the entire history of the earth, which is a function assigning to each time \(t \in \mathbb{R}\) a state \(x(t)\). For example, if we choose coordinates so that the earth’s orbit is in the plane of the first two coordinates, and take the orbit to be a perfect circle, the function describing the earth now is something like

\[
x(t) = ((R \cos(\phi + 2\pi t/T), R \sin(\phi + 2\pi t/R), 0),
\]

\[
(-2\pi R/T) \sin(\phi + 2\pi t/T), (2\pi R/T) \cos(\phi + 2\pi t/T), 0).
\]  

(2.2)

Here

\[
R = \text{distance to the sun} \approx 1.496 \times 10^{11} \text{ meters},
\]

\[
T = \text{length of the year} \approx 3.15569 \times 10^7 \text{ seconds},
\]

and \(\phi\) is the angular position of the earth in its orbit at time zero.

The job of classical physics is to tell you how to find the future \(x(t)\) from knowledge of the state \(x(t_0)\) at one time \(t_0\). Typically the answer is in the form of a differential equation

\[
\frac{dx}{dt} = F(x, t),
\]  

(2.3)

which says that how the state of the system changes depends on its present state and the present time. The right side \(F(x, t)\) is a “direction” at \(x\). If \(X\) is inside a vector space, then \(F(x, t)\) is always a vector in that same space; so \(F(x, t)\) is a vector field on \(X\) (maybe changing with time.) If you know what a manifold is, and \(X\) is a manifold, then \(F(x, t)\) belongs to the tangent space \(T_x(X)\) to \(X\) at \(x\).

For the motion of the earth around the sun, Newton’s laws give the differential equation as

\[
\frac{d(p, v)}{dt} = (v(t), -GM_s \frac{p(t)}{\|p(t)\|^3}).
\]

(The first three coordinates of the equation are just the definition of velocity. What’s interesting is how the velocity is changing.) Here \(G\) is Newton’s gravitational constant, and \(M_s\) is the mass of the sun. (By comparing this equation with the formula for \(x(t)\) above, you can get a value for \(GM_s\).)
A nice feature of this equation is that it depends only on \( x(t) \) and not otherwise on \( t \): the laws of physics are not changing with time.

In addition to predicting the future, physicists like to look at things. A \textit{classical observable} is a real-valued function on the state space \( X \):

\[
A : X \to \mathbb{R}.
\]

If the system is in the state \( x \in X \), making the observation yields the number \( A(x) \). The equation of motion (2.3) and the chain rule tell you how the observable \( A \) changes as the system evolves:

\[
\frac{dA(x(t))}{dt} = (F(*, t) \cdot A)(x(t)).
\] (2.4)

The rate of change of \( A \) is gotten by taking the directional derivative of \( A \) in the direction \( F \) (in abstract mathematical language, applying the vector field \( F \) to \( A \)) then evaluating at \( x(t) \).

\textbf{Definition 2.5.} The observable \( A \) is \textit{conserved} by the classical physical system (2.3) if \( F(*, t) \cdot A = 0 \); that is, if \( A \) is constant in the directions of the vector fields \( F(*, t) \). In this case, the value of \( A \) is constant in \( t \) for any possible history \( x(t) \).

For the motion of the earth around the sun, a typical observable is the distance to the sun:

\[
d(p, v) = \|p\|.
\]

Even though this observable happens to be constant in the circular orbit solution (2.2), it is not conserved: there are other possible histories (like elliptical orbits) in which it is not constant.

### 3 Quantum physics

So here is the corresponding toy picture of quantum mechanics. The underlying mathematical structure of a state space \( X \) is replaced by a \textit{complex inner product space}, often denoted \( \mathcal{H} \). (This is what we have called \( V \) in class.) A \textit{state} of the physical system is a line (a one-dimensional complex subspace) of \( \mathcal{H} \). Another way to say this is that a state is a nonzero vector \( \psi \in \mathcal{H} \), and that \( \psi \) and \( z\psi \) define the same state whenever \( z \) is a nonzero complex number. (The vector \( \psi \) is just like the vectors we’ve been calling \( v \); I switched to the Greek letter only to follow common physics notation.)

A lot of sources say that a state is a vector \( \psi \) with \( \|\psi\| = 1 \) (called a \textit{unit vector}) and that the unit vectors \( \psi \) and \( e^{i\theta}\psi \) define the same state. (In
the Copenhagen Interpretation of quantum mechanics, this corresponds to the idea that no experiment can be designed that will distinguish between the state $\psi$ and the state $e^{i\theta}\psi$.

I’ll stick with the idea that a state is a line $C\psi$, and that the chosen basis vector $\psi$ for the line need not be a unit vector.

Just as in classical mechanics, the state of the system changes in time. In general, this evolution is not described as a changing line, but as a changing basis vector for the line. That is, there is supposed to be a function

$$\psi: \mathbb{R} \rightarrow \mathcal{H} - 0, \quad C \cdot \psi(t) = \text{state at time } t. \quad (3.1)$$

The laws of physics are supposed to be summarized by a self-adjoint operator

$$H \in \mathcal{L}(\mathcal{H}), \quad H^* = H, \quad (3.2)$$

called the Hamiltonian of the physical system. I’ll say more in a moment about the physical interpretation of this linear transformation. The first point is that the quantum-mechanical version of (2.3) is the Schrödinger equation

$$\frac{d\psi(t)}{dt} = \frac{1}{i\hbar}H\psi(t). \quad (3.3)$$

The first big difference from (2.3) is that the Schrödinger equation is required to be linear. This seems to make quantum mechanics simpler than classical mechanics. One reason that quantum mechanics can stay frightening is that the vector space $\mathcal{H}$ is most often infinite-dimensional.

Suppose $\psi(t)$ is a solution of the Schrödinger equation. Let us see how the length of the vector $\psi(t) \in \mathcal{H}$ changes in time. We calculate

$$\frac{d\langle\psi(t),\psi(t)\rangle}{dt} = \frac{d\psi(t)}{dt},\psi(t) + \langle\psi(t),\frac{d\psi(t)}{dt}\rangle \quad \text{(product rule)}$$

$$= \langle \frac{1}{i\hbar}H\psi(t),\psi(t) \rangle + \langle \psi(t),\frac{1}{i\hbar}H\psi(t) \rangle \quad \text{(Schrödinger)}$$

$$= \langle \frac{1}{i\hbar}H\psi(t),\psi(t) \rangle - \langle \frac{1}{i\hbar}\psi(t),H\psi(t) \rangle \quad \text{(Hermitian)}$$

$$= \langle \frac{1}{i\hbar}H\psi(t),\psi(t) \rangle - \langle \frac{1}{i\hbar}H\psi(t),\psi(t) \rangle \quad \text{(selfadjoint)}$$

$$= 0. \quad (3.4)$$

Therefore

a solution $\psi(t)$ to Schrödinger’s equation (3.3) has constant length. \quad (3.5)
A quantum observable is a self-adjoint linear transformation on
\[ A \in \mathcal{L}(\mathcal{H}), \quad A^* = A. \tag{3.6a} \]
If \( \mathcal{H} \) is finite-dimensional, then the spectral theorem says
\[ \mathcal{H} = \bigoplus_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda, \tag{3.6b} \]
with \( \mathcal{H}_\lambda \) the \( \lambda \)-eigenspace of the observable \( A \). (If \( \mathcal{H} \) is infinite-dimensional, there are still versions of the spectral theorem available. The number of eigenvalues may be infinite, and proving the spectral theorem requires more work; but the finite-dimensional case still gives a reasonable picture of what is going on.)

The possible values of the observable \( A \) are the real numbers that are eigenvalues of \( A \). If a state \( \mathcal{C}\psi \) is contained in the eigenspace \( \mathcal{H}_\lambda \), then we say that the value of the observable is \( \lambda \).

The central idea of quantum mechanics is this: most vectors in \( \mathcal{H} \) (and therefore most states of the corresponding physical system) do not belong to any one eigenspace. Instead the vectors (and so the states) are linear combinations (the physics word is superpositions) of many eigenvectors. What this means is that the observable is simultaneously taking on many different values. This isn’t how classical observables work, but quantum mechanics is just different.

To be more concrete, suppose that the observable \( A \) has the spectral decomposition written in (3.6b). If \( \mathcal{C}\psi \) is a state, then the eigenspace decomposition of \( \psi \) is
\[ \psi = \sum_{\lambda \in \mathbb{R}} \psi_\lambda, \quad A\psi_\lambda = \lambda \psi_\lambda. \tag{3.6c} \]
Because the decomposition is orthogonal, the Pythagorean Theorem says
\[ \langle \psi, \psi \rangle = \sum_{\lambda \in \mathbb{R}} \langle \psi_\lambda, \psi_\lambda \rangle, \tag{3.6d} \]
or equivalently
\[ 1 = \sum_{\lambda \in \mathbb{R}} \frac{\langle \psi_\lambda, \psi_\lambda \rangle}{\langle \psi, \psi \rangle}. \tag{3.6e} \]
On the right we have a bunch of nonnegative numbers adding up to one. Whenever you see such a thing, you should think probability. In this case, one of the standard ways of thinking about quantum mechanics is
\[ \frac{\langle \psi_\lambda, \psi_\lambda \rangle}{\langle \psi, \psi \rangle} = \text{probability that observable } A \text{ has value } \lambda. \tag{3.6f} \]
The Copenhagen Interpretation raises this idea a bit higher: it says that if the quantum system is in the state $\psi$, and you perform an experiment to measure the observable $A$, then the probability that you will measure $\lambda$ is equal to $\frac{\langle \lambda \psi, \psi \rangle}{\langle \psi, \psi \rangle}$. Quantum mechanics doesn’t tell you what will happen: it offers a library of possible outcomes (the eigenvalues of $A$) and tells you the probability of seeing each of them.

One of the basic ideas in probability is the idea of expected value. This is the average of all possible outcomes of some trial, with the outcomes weighted by their probability. If I have five test papers in a hat, with the scores 58, 63, 79, 87, and 98, and I draw one paper at random from the hat, then the expected value of the score is

$$\frac{1}{5} \cdot 58 + \frac{1}{5} \cdot 63 + \frac{1}{5} \cdot 79 + \frac{1}{5} \cdot 87 + \frac{1}{5} \cdot 98 = 77,$$

the ordinary average of the scores. If I roll a fair die, the expected number appearing is

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5.$$  

(Notice that this “expected value” cannot actually occur.)

In the Copenhagen Interpretation, the expected value of the experiment to measure the value of the observable $A$ is

$$E(A) = \sum_{\lambda \in \mathbb{R}} \frac{\langle \psi, \lambda \psi \rangle}{\langle \psi, \psi \rangle} \cdot \lambda$$

(weighted average of outcomes)

$$= \sum_{\lambda \in \mathbb{R}} \frac{\langle \lambda \psi, \psi \rangle}{\langle \psi, \psi \rangle}$$

(orthogonality of eigenspaces)

$$= \sum_{\lambda \in \mathbb{R}} \frac{\langle \lambda \psi, \psi \rangle}{\langle \psi, \psi \rangle}$$

(linearity of inner product)

$$E(A) = \frac{\langle A \psi, \psi \rangle}{\langle \psi, \psi \rangle}.$$ 

This expected value varies continuously with the quantum state, from a maximum of the largest eigenvalue of $A$ (if the state belongs to the corresponding eigenspace) to a minimum of the smallest eigenvalue of $A$. 

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It’s natural to try to concentrate on observables that you can reliably observe; that is, to look only at states \( \psi \in \mathcal{H}_\lambda \) that are eigenvalues for \( A \). The essential difficulties of quantum mechanics arise when you are interested in two different observables \( A \) and \( A' \), so that there are two different eigenspace decompositions

\[
\mathcal{H} = \bigoplus_{\lambda \in \mathbb{R}} \mathcal{H}_{\lambda, A}, \quad \mathcal{H} = \bigoplus_{\lambda' \in \mathbb{R}} \mathcal{H}_{\lambda', A'}.
\] (3.6h)

In this situation you might like to concentrate on states in which you can reliably observe both \( A \) and \( A' \); that is, states that are in a “simultaneous eigenspace”

\[
\mathcal{H}_{(\lambda, \lambda')} = \mathcal{H}_{\lambda, A} \cap \mathcal{H}_{\lambda', A'}.
\] (3.6i)

The difficulty is that all these spaces can be zero: there may be no simultaneous eigenspaces of \( A \) and \( A' \). This is not something pathological, but rather entirely typical.

**Example 3.7.** Suppose \( \mathcal{H} = \mathbb{C}^2 \), so that self-adjoint operators are \( 2 \times 2 \) matrices

\[
\begin{pmatrix}
a & z \\
z & b
\end{pmatrix} \quad (a \in \mathbb{R}, z \in \mathbb{C}).
\]

Two natural observables are

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Each of these observables has eigenvalues \(+1\) and \(-1\); each can take just those two values “classically.” (In honest physics, observables like this arise for example in describing the polarization of light.) The spectral decompositions are

\[
\mathcal{H} = \mathcal{H}_{1, A} \oplus \mathcal{H}_{-1, A} = \mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

\[
\mathcal{H} = \mathcal{H}_{1, A'} \oplus \mathcal{H}_{-1, A'} = \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

The eigenspaces for \( A \) are the two coordinate axes; and the eigenspaces for \( A' \) are the two diagonal lines. There are no simultaneous eigenvectors except 0, which does not represent a state (since a quantum state is a one-dimensional subspace).
Problem set 9 says that if the observables $A$ and $A'$ commute, then there is a simultaneous eigenspace decomposition. (It’s easy to see that commuting is a necessary condition for the simultaneous eigenspace decomposition: do you see why?) So quantum-mechanical difficulties are attached to observables that do not commute. Example 3.7 says that this is mathematically very common. One of the fundamental physical principles of quantum mechanics is that the observables position and momentum never commute. That is, the most basic physical observables—position and velocity—can never be reliably observed at the same time. That’s what the Heisenberg uncertainty principle is about. Some details may be found in Section 4 (see (4.5b)).

The Hamiltonian $H$ appearing in the Schrödinger equation (3.3) is a self-adjoint operator, so it is also an observable. It’s the most important observable of all: the energy of the physical system. The eigenvalues of $H$ are the possible energies that the system can have. If $\psi_0 \in \mathcal{H}_E$ is in the $E$-eigenspace—that is, if $\psi_0$ has energy equal to $E$—then $H\psi_0 = E\psi_0$, and we get an easy solution to the Schrödinger equation:

$$\psi(t) = e^{Et/\hbar}\psi_0.$$ (3.8)

That is, a state purely in the energy level $E$ does not change; only its phase oscillates, with a frequency of

$$\nu = 2\pi E/\hbar = E/h.$$ (3.9)

This is Planck’s relation between frequency and energy: originally found for photons, but now making some kind of sense for any quantum-mechanical system.

**Definition 3.10.** The observable $A$ is **conserved** by the quantum physical system (3.3) if the expected value of the observable

$$\frac{\langle A\psi(t), \psi(t) \rangle}{\langle \psi(t), \psi(t) \rangle}$$

is constant in time.

**Proposition 3.11.** Suppose

$$\psi: \mathbb{R} \to \mathcal{H}$$

is a solution of the Schrödinger equation (3.3), and that $A \in \mathcal{L}(\mathcal{H})$ is an observable. Then the expected value of $A$

$$E(A)(t) = \frac{\langle A\psi(t), \psi(t) \rangle}{\langle \psi(t), \psi(t) \rangle}$$

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satisfies the differential equation

\[ \frac{dE(A)}{dt} = E\left(\frac{1}{i\hbar}(AH - HA)\right). \]

Here \( \frac{1}{i\hbar}(AH - HA) \in \mathcal{L}(\mathcal{H}) \) is a self-adjoint linear transformation (and so a new observable).

In particular, \( A \) is conserved if and only if \( AH = HA \); that is, if and only if the linear maps \( A \) and \( H \) commute with each other.

To a mathematician, the proposition above suggests that one should replace each physical observable

\[ A = \text{selfadjoint operator} \quad (A^* = A) \quad (3.12a) \]

by a “mathematical observable”

\[ S_A = \text{skew-adjoint operator} \quad (S^* = -S), \quad (3.12b) \]

according to the rule

\[ S_A = \frac{1}{i\hbar}A, \quad A = i\hbar S. \quad (3.12c) \]

Given two mathematical observables \( S \) and \( S' \), one can form a new mathematical observable

\[ [S, S'] = SS' - S'S, \quad (3.12d) \]

called the commutator of \( S \) and \( S' \). This operation, called commutator or Lie bracket makes the mathematical observables into a Lie algebra. The Schrödinger equation takes the form

\[ \frac{d\psi(t)}{dt} = S_H\psi(t). \quad (3.12e) \]

The differential equation in the proposition above is

\[ \frac{dE(A)}{dt} = E(i\hbar[S_A, S_H]). \quad (3.12f) \]

So the commutator describes exactly the how observables change in time.
4 Harmonic oscillator

This possible future section will write down the classical and quantum models for a one-dimensional harmonic oscillator, and compare their solutions. For now I’ll just write a little. The inner product space is

\[ \mathcal{H} = L^2(\mathbb{R}) = \{ \psi: \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx < \infty \}. \]  

(4.1)

We’ll be vague about exactly which functions \( \psi \) are allowed; this is one of the places where the details are difficult.

The inner product is like the one used on functions on \([0,1]\) in the text and problem sets:

\[ \langle \psi_1, \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1(x) \overline{\psi_2(x)} \, dx. \]  

(4.2)

Recall that a state corresponds to a line \( \mathbb{C} \psi \subset \mathcal{H} \). We think of it as corresponding to a (quantum) particle living at some indeterminate place on the real line; the size \( |\psi(x)|^2 \) of the function represents the probability that the particle is at \( x \). (A little more precisely, the integral

\[ \frac{1}{\langle \psi, \psi \rangle} \int_a^b |\psi(x)|^2 \, dx \]

is the probability that the particle is between \( a \) and \( b \).)

The classical observable position corresponds to the (selfadjoint) quantum observable

\[ (M \psi)(x) = x \psi(x). \]  

(4.3a)

That is, we multiply the function \( \psi \) by the function \( x \). (The letter \( M \) is chosen to be a reminder of “multiply.”) The eigenspace for a real number \( \lambda \) is

\[ \mathcal{H}_{\lambda,M} = \mathcal{H}_{\lambda} = \{ \psi \mid x \psi(x) = \lambda \psi(x) \} \]

\[ = \{ \psi \mid \psi(x') = 0, x' \neq \lambda \}. \]  

(4.3b)

That is, the eigenspace consists of multiples of the Dirac delta “function”

\[ \delta_{\lambda}, \quad \delta_{\lambda}(x) = \begin{cases} \infty & x = \lambda, \\ 0 & x \neq \lambda. \end{cases} \]  

(4.3c)

The “function” \( \delta_{\lambda} \) is not in \( L^2(\mathbb{R}) \), because it is too concentrated at the point \( \lambda \). (I can’t make this precise, because I haven’t said exactly what functions are allowed in \( L^2(\mathbb{R}) \). The idea is that a nice function on the real line that is
zero except at one point must be zero everywhere.) Nevertheless, one should think of every real number \( \lambda \) as an eigenvalue of \( M \), with eigenvector \( \delta_\lambda \).

Even though it is not an actual function, certain integrals involving \( \delta_\lambda \) make sense. What is required (and what becomes the mathematical definition of \( \delta_\lambda \)) is that for any nice function \( \phi \),

\[
\int_{-\infty}^{\infty} \phi(x) \delta_\lambda(x) \, dx = \phi(\lambda) \tag{4.3d}
\]

Using the formal “definition” \( (4.3c) \), it is clear that

\[
\delta_\lambda(x) = \delta_x(\lambda). \tag{4.3e}
\]

Using the mathematical definition hinted at in \( (4.3d) \), it is possible to make this statement meaningful and precise; but I won’t worry about it.

In the case of a self-adjoint operator \( A \) on a finite-dimensional vector space \( V \), we can choose a basis of eigenvectors

\[
(e_{\lambda_1}, \ldots, e_{\lambda_n}), \quad Ae_{\lambda_i} = \lambda_i e_{\lambda_i}
\]

Then any vector in \( v \) has a finite sum expansion \( v = \sum_i a_i e_{\lambda_i} \); the magnitude \( |a_i|^2 \) tells how much of \( v \) is in the direction of the eigenvalue \( \lambda_i \).

In this infinite-dimensional, setting, the finite sum is replaced by an integral

\[
\phi = \int_{-\infty}^{\infty} a(\lambda) \delta_\lambda \, d\lambda \tag{4.3f}
\]

In order to see what the coefficients \( a(\lambda) \) of the various eigenvectors \( \delta_\lambda \) should be, first plug in \( x \); then use the formula \( (4.3e) \); and finally use \( (4.3d) \):

\[
\phi(x) = \int_{-\infty}^{\infty} a(\lambda) \delta_\lambda(x) \, d\lambda = \int_{-\infty}^{\infty} a(\lambda) \delta_x(\lambda) \, d\lambda = a(x). \tag{4.3g}
\]

The conclusion is that

\[
a(x) = \phi(x), \quad \phi = \int_{-\infty}^{\infty} \phi(\lambda) \delta_\lambda \, d\lambda. \tag{4.3h}
\]

That is, the size of \( |\phi(\lambda)|^2 \) measures how much of the state \( \phi \) is at the position \( \lambda \).
The classical observable \textit{momentum} corresponds to the (selfadjoint) quantum observable
\[(D\psi)(x) = -i\hbar \frac{d\psi}{dx},\] (4.4a)
a multiple of the derivative of \(\psi\). (The letter \(D\) is chosen to be a reminder of “derivative.”) The eigenspace for a real number \(\mu\) is
\[\mathcal{H}_{\mu,D} = \mathcal{H}_\mu \equiv \{ \psi \mid D\psi(x) = \mu\psi(x) \} \]
\[= \{ \psi(x) = Ae^{i\mu x/\hbar} \}.\] (4.4b)
That is, the \(\mu\) eigenspace consists of multiples of the exponential function
\[e_{\mu/\hbar}, \quad e_\xi(x) \equiv e^{2\pi i \xi x}.\] (4.4c)
These exponential functions are not in \(L^2(\mathbb{R})\), because they are too spread out over \(\mathbb{R}\): the absolute value squared is one everywhere, so cannot have finite integral. Nevertheless, one should think of every real number \(\mu\) as an eigenvalue of \(D\), with eigenvector \(e_{\mu/\hbar}\).

Just as in the case of position, we want to replace the finite sum in the finite-dimensional spectral theorem by an integral, and write any vector \(\phi\) as an integral of eigenvectors:
\[\phi = \int_{-\infty}^{\infty} b(\mu)e_{\mu/\hbar} \, d\mu\] (4.4d)
In order to see what the coefficients \(b(\mu)\) of the various momentum eigenvectors \(e_{\mu/\hbar}\) should be, first plug in \(x\), then perform the change of variable \(\xi = \mu/\hbar\):
\[\phi(x) = \int_{-\infty}^{\infty} b(\mu)e_{\mu/\hbar} \, d\mu\]
\[= \frac{1}{\hbar} \int_{-\infty}^{\infty} b(h\xi)e_{\xi} \, d\xi\] (4.4e)

The integral on the left is an \textit{inverse Fourier transform}. Here are the basic definitions and facts. If \(\phi\) and \(\psi\) are nice complex-valued functions on \(\mathbb{R}\), then the \textit{Fourier transform} of \(\phi\) is a new (nice) complex-valued function on \(\mathbb{R}\), defined by
\[\hat{\psi}(\xi) = \int_{-\infty}^{\infty} \psi(x)e^{-\xi x} \, dx = \int_{-\infty}^{\infty} \psi(x)e^{-2\pi i x\xi} \, dx.\] (4.4f)
The \textit{inverse Fourier transform} of \(\phi\) is the new (nice) complex-valued function on \(\mathbb{R}\) defined by
\[\check{\phi}(x) = \int_{-\infty}^{\infty} \phi(\xi)e_{\xi}(x) \, d\xi = \int_{-\infty}^{\infty} \phi(\xi)e^{2\pi i x\xi} \, d\xi.\] (4.4g)
The main theorem (Fourier inversion) is that the Fourier transform is an isometry (length-preserving invertible linear map) on $L^2(\mathbb{R})$, with inverse equal to the inverse Fourier transform:

$$\hat{(\phi)^*} = \phi, \quad \|\hat{\phi}\| = \|\phi\| \quad (\phi \in L^2(\mathbb{R})).$$ (4.4h)

Comparing (4.4e) with (4.4g), we conclude that

$$b(\mu) = \hat{\phi}(\mu/h), \quad \phi = \frac{1}{\hbar} \int_{-\infty}^{\infty} \hat{\phi}(\mu/h)e^{\mu/h} d\mu.$$ (4.4i)

That is, the size of $|\hat{\phi}(\mu/h)|^2$ measures how much of the state $\phi$ has momentum $\mu$.

The main point of all this is that the states corresponding to a precise position $\lambda$—the “Dirac delta functions” $\delta_\lambda$—are completely different from the states corresponding to a precise momentum $\mu$—the complex exponentials $e^{\mu/h}$. We saw in a pset that commuting selfadjoint operators could be simultaneously diagonalized: that there would in lots of simultaneous eigenstates, where both observables are precisely known. The position and momentum operators $P$ and $D$ do not commute; in fact

$$\begin{align*}
(MD - DM)\phi &= -ix\hbar \frac{d\phi}{dx} + i\hbar \frac{d}{dx}(x\phi) \\
&= -ix\hbar \frac{d\phi}{dx} + i\hbar (x\frac{d\phi}{dx} + \phi) \\
&= i\hbar \phi.
\end{align*}$$ (4.5a)

That is,

$$[M, D] = i\hbar,$$ (4.5b)

the canonical commutation relation of Heisenberg. If we use the language of skew-adjoint “mathematical observables” discussed in (3.12), then the skew-adjoint operators corresponding to position and momentum are

$$S_M = x/i\hbar, \quad S_D = -\frac{d}{dx}, \quad [S_M, S_D] = \frac{1}{i\hbar}.$$ (4.5c)

As explained in the text, no such commutation relations can be satisfied by operators on a finite-dimensional inner product space: the left side has trace equal to zero, and the right side has trace equal to $(i\hbar)^{\pm 1}$ times the dimension of the space. So any quantum mechanics involving positions and momenta has to live in an infinite-dimensional inner product space.

So you need to take more math classes after this one!