18.700 Problem Set 6

Due in class Monday December 4; late work will not be accepted. Your work on graded problem sets should be entirely your own.

1. This problems concerns the 3×3 real matrix

$$A = \begin{pmatrix} 1/2 & 1/4 & 1/6 \\ 1/3 & 1/2 & 1/3 \\ 1/6 & 1/4 & 1/2 \end{pmatrix}.$$

a) (5 points) Define a subspace of $\mathbb{R}^3$ by

$$W = \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right).$$

Prove that any vector in W must have the sum of its coordinates equal to zero.

b) (10 points) Find the eigenvalues and eigenvectors of A. (Two hints: 1 is an eigenvalue, and part (a) is not completely unrelated to this problem.)

c) (5 points) Suppose v is a vector in $\mathbb{R}^3$ such that the sum of the coordinates of v is 1. Prove that

$$\lim_{n \to \infty} A^n v = \begin{pmatrix} 3/10 \\ 4/10 \\ 3/10 \end{pmatrix}.$$

(Hint: you can write v as a linear combination of eigenvectors of A.)

2. (5 points) The two vectors

$$w_1 = (1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3}), \qquad w_2 = (1/\sqrt{2},\ 0,\ -1/\sqrt{2})$$

are orthogonal vectors of length 1 in $\mathbb{R}^3$. Find a third vector w_3 so that $\{w_1, w_2, w_3\}$ is an orthonormal basis.

3. (5 points) Suppose that V is a real inner product space, and u and v are vectors in V. Prove the *polar identity*

$$\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2).$$

(One consequence of this identity is that any linear transformation of V that preserves lengths of vectors must also preserve inner products.)