18.700 Problem Set 4

Due in class Friday November 3; late work will not be accepted. Your work on graded problem sets should be entirely your own.

A few of you may notice that the points assigned here add up to 35 instead of 30. You can think of this as extra credit bestowed by a generous professor, or as extra work inflicted by whatever. You can even (in either interpretation) omit one of the seven parts.

1. (20 points) Let V be the real vector space consisting of all functions on the real line of the form $p(x)e^{-x^2}$, with p a polynomial of degree at most 3.

a) Find a basis for V, and therefore compute the dimension of V.

b) If $f \in V$, define a new function Tf by

$$Tf = \frac{d^2 f}{dx^2} - (4x^2 - 10)f.$$

Prove that T is a linear transformation from V to V. (Hint: the main point is to show that $Tf \in V$.)

c) Find the matrix of T in the basis you gave in (a).

d) Find a basis for $\ker T$.

2. (15 points) Let U, V, and W be finite-dimensional vector spaces, and let

$$T: U \to V, \qquad S: V \to W$$

be linear transformations.

a) Prove that $\operatorname{rank}(S \circ T) \leq \operatorname{rank} S$.

b) Prove that $\operatorname{rank}(S \circ T) \leq \operatorname{rank} T$.

c) Give an example in which it is *not* true that $\operatorname{rank}(S \circ T)$ is equal to the smaller of $\operatorname{rank} S$ and $\operatorname{rank} T$.