18.700 Problem Set 3

Due in class Monday October 23; late work will not be accepted. Your work on graded problem sets should be entirely your own.

1. (10 points) Suppose a, b, c, and d are elements of a field $\mathbb{F}$. Use elementary row and column operations to calculate

$$\det \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}.$$

Under what conditions on a, b, c, and d is this determinant non-zero? (As a check on your work, the answer to this last question is fairly simple.)

2. (10 points) Suppose the field $\mathbb{F}$ is $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. You can think of choosing a vector in $\mathbb{F}^n$ "at random" by flipping a coin n times, and putting a 0 for each coordinate where you get a heads, and a 1 where you get tails. In the same way, you can make a random $n \times n$ matrix by flipping n^2 times. Each of the 2^{n^2} matrices has an equal chance to appear. The determinant of such a matrix is 0 or 1. This problem looks at which determinant is more likely; that is, at whether a random matrix is "likely" to be invertible.

a) How many of the 16 2×2 matrices have determinant 1?

b) (This one is hard.) Suppose that A is an $(n-1) \times n$ matrix (over $\mathbb{Z}/2\mathbb{Z}$) of rank exactly $n-1$. Show that there are exactly 2^{n-1} row vectors $v \in \mathbb{F}^n$ with the property that the $n \times n$ matrix $\begin{pmatrix} A \\ v \end{pmatrix}$ is invertible.

c) Explain why less than half of all $n \times n$ matrices are invertible.

3. (10 points) This problem is about square matrices A with integer entries, whose inverses also have integer entries. The only 1×1 matrices of this type are (± 1)

a) Suppose $A = \begin{pmatrix} p & n \\ 0 & q \end{pmatrix}$, with p, q, and n integers. Show that A^{-1} has integer entries if and only if p and q are both ± 1.

b) Suppose $A = \begin{pmatrix} n & p \\ q & 0 \end{pmatrix}$, with p, q, and n integers. Show that A^{-1} has integer entries if and only if p and q are both ± 1.

c) Give an example of a 2×2 matrix A with integer entries, *none* of which is ± 1, such that A^{-1} exists and has integer entries.