Coherent sheaves on nilpotent cones

David Vogan

Department of Mathematics
Massachusetts Institute of Technology

Taipei Conference in Representation Theory V
Academica Sinica January 2016
Outline

Introduction

What are the questions?

Equivariant $K$-theory

$K$-theory and representations

Complex groups: $\infty$-diml reps and algebraic geometry

Lusztig’s conjecture and generalizations

Slides at \url{http://www-math.mit.edu/~dav/paper.html}
Why coherent sheaves?

Compact groups $K$ are relatively easy. . .
Noncompact groups $G$ are relatively hard.
Harish-Chandra et al. idea:

understand $\pi \in \hat{G} \iff$ understand $\pi|_K$

(nice compact subgroup $K \subset G$).

Get an invariant of a repn $\pi \in \hat{G}$:

$$m_\pi : \hat{K} \to \mathbb{N}, \quad m_\pi(\mu) = \text{mult of } \mu \text{ in } \pi|_K.$$ 

1. What’s the support of $m_\pi$? (subset of $\hat{K}$)
2. What’s the rate of growth of $m_\pi$?
3. What functions on $\hat{K}$ can be $m_\pi$?

Answers $\iff$ sheaves on nilpotent cones.
Examples

1. $G = GL(n, \mathbb{C}), K = U(n)$. Typical restriction to $K$ is
   \[
   \pi|_K = \text{Ind}^{U(n)}_{U(1)^n}(\gamma) = \sum_{\mu \in \widehat{U}(n)} m_\mu(\gamma) \gamma \quad (\gamma \in \widehat{U(1)^n}) ;
   \]
   \[m_\pi(\mu) = \text{mult of } \mu \text{ is } m_\mu(\gamma) = \text{dim of } \gamma \text{ wt space}.
   \]
2. $G = GL(n, \mathbb{R}), K = O(n)$. Typical restriction to $K$ is
   \[
   \pi|_K = \text{Ind}^{O(n)}_{O(1)^n}(\gamma) = \sum_{\mu \in \widehat{O(n)}} m_\mu(\gamma) ;
   \]
   \[m_\pi(\mu) = \text{mult of } \mu \text{ in } \pi \text{ is } m_\mu(\gamma) = \text{mult of } \gamma \text{ in } \mu.
   \]
3. $G$ split of type $E_8$, $K = \text{Spin}(16)$. Typical res to $K$ is
   \[
   \pi|_{\text{Spin}(16)} = \text{Ind}^{\text{Spin}(16)}_M(\gamma) = \sum_{\mu \in \widehat{\text{Spin}(16)}} m_\mu(\gamma) \gamma ;
   \]
   here $M \subset \text{Spin}(16)$ subgp of order 512, cent ext of $(\mathbb{Z}/2\mathbb{Z})^8$.

Moral: may compute $m_\pi$ using compact groups.
Plan for today

Work with real reductive Lie group $G(\mathbb{R})$.
Describe (old) assoc cycle $\mathcal{AC}(\pi)$ for $\pi \in \hat{G}(\mathbb{R})$:

$\approx$ geom shorthand for approximating $\pi|_{K(\mathbb{R})}$.

Describe (new) algorithm for computing $\mathcal{AC}(\pi)$.

A real algorithm is one that’s been implemented on a computer. This one has not, but should be possible soon.
Assumptions

\[ G(\mathbb{C}) = G = \text{cplx conn reductive alg gp.} \]
\[ G(\mathbb{R}) = \text{group of real points for a real form.} \]
Could allow \text{fin cover of open subgp of } G(\mathbb{R}), \text{so allow nonlinear.}
\[ K(\mathbb{R}) \subset G(\mathbb{R}) \text{ max cpt subgp; } K(\mathbb{R}) = G(\mathbb{R})^\theta. \]
\[ \theta = \text{alg inv of } G; K = G^\theta \text{ possibly disconn reductive.} \]

Harish-Chandra idea:
\[
\begin{align*}
\infty\text{-diml reps of } G(\mathbb{R}) & \leftrightarrow \text{alg gp } K \curvearrowright \text{cplx Lie alg } g \\
(g, K)\text{-module} & \text{ is vector space } V \text{ with} \\
1. & \text{ repn } \pi_K \text{ of algebraic group } K: V = \sum_{\mu \in \hat{K}} m_V(\mu) \mu \\
2. & \text{ repn } \pi_g \text{ of cplx Lie algebra } g \\
3. & d\pi_K = \pi_g|_t, \quad \pi_K(k)\pi_g(X)\pi_K(k^{-1}) = \pi_g(\text{Ad}(k)X).
\end{align*}
\]
In module notation, cond (3) reads \[ k \cdot (X \cdot v) = (\text{Ad}(k)X) \cdot (k \cdot v). \]
Geometrizing representations

\[ G(\mathbb{R}) \text{ real reductive, } K(\mathbb{R}) \text{ max cpt, } \mathfrak{g}(\mathbb{R}) \text{ Lie alg} \]
\[ \leadsto K \text{ cplx reductive alg gp } \hookrightarrow \mathfrak{g} \text{ cplx reduc Lie alg.} \]
\[ \mathcal{N}^* = \text{cone of nilpotent elements in } \mathfrak{g}^*. \]
\[ \mathcal{N}^*_\theta = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*, \text{ finite # nilpotent } K \text{ orbits.} \]

**Goal 1:** Attach nilp orbits to repns in theory.

**Goal 2:** Compute them in practice.

"In theory there is no difference between theory and practice. In practice there is." Jan L. A. van de Snepscheut (or not).

\[ V \text{ irr } \mathfrak{g}, K \text{-module} \]
\[ \downarrow \text{ assoc cycle of gr} \]
\[ \mathcal{AC}(V) \text{ closed union of } K \text{ orbits on } \mathcal{N}^*_\theta \]

So **Goal 1** is completed. Turn to **Goal 2**...
Associated varieties

\[ \mathcal{F}(g, K) = \text{finite length } (g, K)\text{-modules} \ldots \]

noncommutative world we care about.

\[ \mathcal{C}(g, K) = \text{f.g. } (S(g/\mathfrak{t}), K)\text{-modules, support } \subset \mathcal{N}_\theta^* \ldots \]

commutative world where geometry can help.

\[ \mathcal{F}(g, K) \xrightarrow{\text{gr}} \mathcal{C}(g, K) \]

**Prop.** gr induces surjection of Grothendieck groups

\[ K\mathcal{F}(g, K) \xrightarrow{\text{gr}} K\mathcal{C}(g, K); \]

image records restriction to \( K \) of HC module.

So restrictions to \( K \) of HC modules sit in equivariant coherent sheaves on nilpotent cone in \((g/\mathfrak{t})^*\)

\[ KC(g, K) =_{\text{def}} K^K(\mathcal{N}_\theta^*), \]

equivariant K-theory of the \( K \)-nilpotent cone.

**Goal 2:** compute \( K^K(\mathcal{N}_\theta^*) \) and the map **Prop.**
Equivariant $K$-theory

**Setting:** (complex) algebraic group $K$ acts on (complex) algebraic variety $X$.

$\text{Coh}^K(X) =$ abelian categ of coh sheaves on $X$ with $K$ action.

$K^K(X) = \text{def}$ Grothendieck group of $\text{Coh}^K(X)$.

**Example:** $\text{Coh}^K(\text{pt}) = \text{Rep}(K)$ (fin-diml reps of $K$).

$K^K(\text{pt}) = \mathcal{R}(K) =$ rep ring of $K$; free $\mathbb{Z}$-module, basis $\hat{K}$.

**Example:** $X = K/H$; $\text{Coh}^K(K/H) \cong \text{Rep}(H)$

$E \in \text{Rep}(H) \mapsto \mathcal{E} = \text{def } K \times_H E$ eqvt vector bdle on $K/H$

$K^K(K/H) = \mathcal{R}(H)$.

**Example:** $X = V$ vector space (repn of $K$).

$E \in \text{Rep}(K) \mapsto \text{proj module } \mathcal{O}_V(E) = \text{def } \mathcal{O}_V \otimes E \in \text{Coh}^K(X)$

proj resolutions $\implies K^K(V) \cong \mathcal{R}(K)$, basis $\{\mathcal{O}_V(\tau)\}$. 
Doing nothing carefully

Suppose $K \curvearrowright X$ with finitely many orbits:

$X = Y_1 \cup \cdots \cup Y_r, \quad Y_i = K \cdot y_i \cong K/Ky_i$.

Orbits partially ordered by $Y_i \geq Y_j$ if $Y_j \subset Y_i$.

$(\tau, E) \in \overline{K^y_i} \mapsto \mathcal{E}(\tau) \in \text{Coh}^K(Y_i)$.

Choose (always possible) $K$-equivariant coherent extension

$\widetilde{\mathcal{E}}(\tau) \in \text{Coh}^K(\overline{Y}_i) \mapsto [\widetilde{\mathcal{E}}] \in K^K(\overline{Y}_i)$.

Class $[\widetilde{\mathcal{E}}]$ on $\overline{Y}_i$ unique modulo $K^K(\partial Y_i)$.

Set of all $[\widetilde{\mathcal{E}}(\tau)]$ (as $Y_i$ and $\tau$ vary) is basis of $K^K(X)$.

Suppose $M \in \text{Coh}^K(X)$; write class of $M$ in this basis

$$[M] = \sum_{i=1}^{r} \sum_{\tau \in K^y_i} n_{\tau}(M)[\widetilde{\mathcal{E}}(\tau)].$$

Maximal orbits in $\text{Supp}(M) = \text{maxl } Y_i$ with some $n_{\tau}(M) \neq 0$.

Coeffs $n_{\tau}(M)$ on maximal $Y_i$ ind of choices of exts $\widetilde{\mathcal{E}}(\tau)$. 

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Introduction

Questions

$K$-theory

$K$-theory & repns

Complex groups

Lusztig conjecture
Our story so far

We have found

1. homomorphism
   \[ \text{virt } G(\mathbb{R}) \text{ reps } K\mathcal{F}(g, K) \xrightarrow{\text{gr}} K^K(\mathcal{N}_\theta^*) \text{ eqvt } K\text{-theory} \]

2. geometric basis \( \{[E(\tau)]\} \) for \( K^K(\mathcal{N}_\theta^*) \), indexed by irr reps of isotropy gps

3. expression of \( [\text{gr}(\pi)] \) in geom basis \( \leadsto \mathcal{AC}(\pi) \).

Problem is computing such expressions…

Teaser for the next section: Kazhdan and Lusztig taught us how to express \( \pi \) using std reps \( I(\gamma) \):

\[
[\pi] = \sum_{\gamma} m_\gamma(\pi)[I(\gamma)], \quad m_\gamma(\pi) \in \mathbb{Z}.
\]

\( \{[\text{gr } I(\gamma)]\} \) is another basis of \( K^K(\mathcal{N}_\theta^*) \).

Last goal is compute chg of basis matrix: to write

\[
[E(\tau)] = \sum_{\gamma} n_\gamma(\tau)[\text{gr } I(\gamma)].
\]
The last goal

Studying cone $\mathcal{N}_\theta^* = \text{nilp lin functionals on } g/\mathfrak{k}$.

Found (for free) basis $\{[\mathcal{E}(\tau)]\}$ for $K^K(\mathcal{N}_\theta^*)$, indexed by orbit $K/K^i$ and irr rep $\tau$ of $K^i$.

Found (by rep theory) second basis $\{[\text{gr } I(\gamma)]\}$, indexed by (parameters for) std reps of $G(\mathbb{R})$.

To compute associated cycles, enough to write

$$[\text{gr } I(\gamma)] = \sum_{\text{orbits}} \sum_{\tau \text{ irr for isotropy}} N_\tau(\gamma)[\tilde{\mathcal{E}}(\tau)].$$

Equivalent to compute inverse matrix

$$[\tilde{\mathcal{E}}(\tau)] = \sum_{\gamma} n_\gamma(\tau)[\text{gr } I(\gamma)].$$

Need to relate

geom of nilp cone $\leftrightarrow$ geom of std reps.

Use parabolic subgps and Springer resolution.
Introducing Springer

\[ g = \mathfrak{t} \oplus \mathfrak{s} \] Cartan decomp, \( \mathcal{N}_\theta^* \cong \mathcal{N}_\theta = \text{def } \mathcal{N} \cap \mathfrak{s} \) nilp cone in \( \mathfrak{s} \).

Kostant-Rallis, Jacobson-Morozov: nilp \( X \in \mathfrak{s} \leadsto Y \in \mathfrak{s} \), \( H \in \mathfrak{t} \)

\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,
\]

\[ g[k] = \mathfrak{t}[k] \oplus \mathfrak{s}[k] \] (ad(\( H \)) eigenspace).

\[ \leadsto g[\geq 0] = \text{def } g = I + u \] \( \theta \)-stable parabolic.

**Theorem** (Kostant-Rallis) Write \( O = K \cdot X \subset \mathcal{N}_\theta \).

1. \( \mu : O_Q = \text{def } K \times_{Q \cap K} \mathfrak{s}[\geq 2] \to \overline{O}, \quad (k, Z) \mapsto \text{Ad}(k)Z \) is proper birational map onto \( \overline{O} \).

2. \( K^X = (Q \cap K)^X = (L \cap K)^X(U \cap K)^X \) is a Levi decomp; so \( \hat{K}^X = \text{[(L \cap K)^X]} \).

So have resolution of singularities of \( \overline{O} \):

\[
\begin{array}{ccc}
\text{vec bdle} & \xleftarrow{K \times_{Q \cap K} \mathfrak{s}[\geq 2]} & K/Q \cap K \\
\mu & & \overline{O}
\end{array}
\]

Use it (i.e., copy McGovern, Achar) to calculate equivariant \( K \)-theory...
Using Springer to calculate $K$-theory

$X \in \mathcal{N}_\theta$ represents $\mathcal{O} = K \cdot X$.

$\mu : \mathcal{O}_Q = K \times_{Q \cap K} \mathbb{Z}[\geq 2] \rightarrow \mathcal{O}$ Springer resolution.

**Theorem** Recall $\hat{K}^X = [(L \cap K)^X]$.

1. $K^K(\mathcal{O}_Q)$ has basis of eqvt vec bdles:

$$ (\sigma, F) \in \text{Rep}(L \cap K) \mapsto F(\sigma) .$$

2. Get extension of $\mathcal{E}(\sigma|_{(L \cap K)^X})$ from $\mathcal{O}$ to $\mathcal{O}$

$$ [\overline{F}(\sigma)] = \text{def} \sum_i (-1)^i [R^i\mu_*(F(\sigma))] \in K^K(\mathcal{O}) .$$

3. Compute (very easily) $[\overline{F}(\sigma)] = \sum_{\gamma} n_{\gamma}(\sigma)[\text{gr } I(\gamma)]$.

4. Each irr $\tau \in [(L \cap K)^X]$ extends to (virtual) rep $\sigma(\tau)$ of $L \cap K$; can choose $\overline{F}(\sigma(\tau))$ as extension of $\mathcal{E}(\tau)$.
Now we can compute associated cycles

Recall $X \in N_\theta \leadsto O = K \cdot X$; $\tau \in [(L \cap K)^X]^\sim$. We now have explicitly computable formulas

$$[\widetilde{E}(\tau)] = [\overline{F(\sigma(\tau))}] = \sum_{\gamma} n_\gamma(\tau)[\text{gr} I(\gamma)].$$

Here’s why this does what we want:

1. inverting matrix $n_\gamma(\tau) \leadsto$ matrix $N_\tau(\gamma)$ writing $[\text{gr} I(\gamma)]$ in terms of $[\widetilde{E}(\tau)]$.

2. multiplying $N_\tau(\gamma)$ by Kazhdan-Lusztig matrix $m_\gamma(\pi) \leadsto$ matrix $n_\tau(\pi)$ writing $[\text{gr} \pi]$ in terms of $[\widetilde{E}(\tau)]$.

3. Nonzero entries $n_\tau(\pi) \leadsto AC(\pi)$.

Side benefit: algorithm for $G(\mathbb{R})$ cplx also computes a bijection (conj Lusztig, proof Bezrukavnikov)

$$(\text{dom wts}) \leftrightarrow (\text{pairs } (O, \tau))\ldots$$
Complex groups regarded as real

\[ G_1 = \text{cplx conn reductive alg gp } \iff \text{old } G(\mathbb{R}) \]  
\[ \sigma_1 = \text{cplx conj for compact real form of } G_1. \]  
\[ G = G_1 \times G_1 \text{ complexification of } G_1 \ldots \]

1. \[ \sigma(x, y) = (\sigma_1(y), \sigma_1(x)) \text{ cplx conj for real form } G_1: \]
   \[ G(\mathbb{R}) = G^\sigma = \{ (x, \sigma_1(x) | x \in G_1 \} \cong G_1. \]

2. \[ \theta(x, y) = (y, x) \text{ Cartan inv: } K = G^\theta = (G_1)_\Delta. \]

\[ K\text{-nilp cone } \mathcal{N}_\theta^* \subset g^* \cong G_1\text{-nilp cone } \mathcal{N}_1^* \subset g_1^*. \]
\[ H_1 \subset G_1, \ H = H_1 \times H_1 \subset G, \ T = (H_1)_\Delta \subset K \text{ max tori.} \]
\[ a = \mathfrak{h}^{-\theta} = \{ (Z, -Z) | Z \in \mathfrak{h}_1 \} \text{ Cartan subspace.} \]

Param for princ series rep is \( \gamma = (\lambda, \nu) \in X^*(T) \times a^*: \)

1. \[ l(\lambda, \nu)|_K \cong \text{Ind}^K_T(\lambda); \]
2. \[ \text{virt rep } [l(w_1 \cdot \lambda, w_1 \cdot \nu)] \text{ indep of } w_1 \in W_1; \]
3. \[ \text{gr } l(\lambda, \nu) \in K^K(\mathcal{N}_\theta^*) \cong K^{G_1}(\mathcal{N}_1^*) \text{ indep of } \nu. \]

Conclusion: the set of all \[ \text{gr } l(\lambda) \cong \text{Ind}^K_T(\lambda) \]
(\( \lambda \in X^*(T) \text{ dom} \)) is basis for (virt HC-mods of \( G_1 \)|_K).
Connection with Weyl char formula

\( K \cong G_1 \) cplx conn reductive alg, \( T \cong H_1 \) max torus.

Asserted “\( \{ \text{Ind}^K_T(\lambda) \} \) basis for (virt HC-mods of \( G_1 \))|_K.”

What’s that mean or tell you?

Fix \((F, \mu) \in \hat{K}\) of highest weight \(\mu \in X^\text{dom}(T)\).

\((F, \mu)\) also irr (fin diml) HC-mod for \(G_1\); \((F, \mu)|_K = (F, \mu)\).

Assertion means \( F = \sum_{\gamma \in X^\text{dom}(T)} m_{\gamma}(F) \text{Ind}^K_T(\gamma) \).

Such a formula is a version of Weyl char formula:

\[
(F, \mu) = \sum_{w \in W(K, T)} (-1)^{\ell(w)} \text{Ind}^K_T(\mu + \rho - w\rho)
\]

\[
= \sum_{B \subset \Delta^+} (-1)^{|\Delta^+| - |B|} \text{Ind}^K_T(\mu + 2\rho - 2\rho(B)).
\]

One meaning: if \((E, \gamma) \in \hat{K}\), then

\[
\sum_{w \in W} (-1)^{\ell(w)} m_{E, \gamma}(\mu + \rho - w \cdot \rho) = \begin{cases} 
1 & (\gamma = \mu) \\
0 & (\gamma \neq \mu).
\end{cases}
\]
Lusztig’s conjecture

\[ G \supset B \supset H \text{ complex reductive algebraic.} \]
\[ X^*(H) \supset X^\text{dom}(H) \text{ dominant weights.} \]
\[ \mathcal{N}^* = \text{cone of nilpotent elements in } \mathfrak{g}^*. \]

Lusztig conjecture: there’s a bijection

\[ X^\text{dom} \leftrightarrow \text{pairs } (\xi, \tau) / G \text{ conjugation; } \]
\[ \xi \in \mathcal{N}^*, \tau \in \widehat{G}^\xi \leftrightarrow \text{eqvt vec bdle } \mathcal{E}(\tau) = G \times_{G^\xi} \tau \]

Thm (Bezrukavnikov). There is a preferred virt extension \( \widetilde{\mathcal{E}}(\tau) \) to \( G \cdot \xi \) so

\[
[\widetilde{\mathcal{E}}(\tau)] = \pm [\text{gr } I(\lambda(\xi, \tau))] + \sum_{\gamma < \lambda(\xi, \tau)} n_\gamma(\xi, \tau)[\text{gr } I(\gamma)].
\]

Upper triangularity characterizes Lusztig bijection.
Calculating Lusztig’s bijection

Proceed by upward induction on nilpotent orbit.

Start with \((\xi, \tau)\), \(\xi \in \mathcal{N}^*\), \(\tau \in \overline{G^\xi}\).

JM parabolic \(Q = LU\), \(\xi \in (\mathfrak{g}/\mathfrak{q})^*\); \(G^\xi = Q^\xi = L^\xi U^\xi\).

Choose virt rep \([\sigma(\tau)] \in R(L)\) extension of \(\tau\).

Write formula for corr ext of \(\mathcal{E}(\tau)\) to \(\overline{G \cdot \xi}\):

\[
[F(\sigma(\tau))] = \sum_{\lambda} m_{\sigma(\tau)}(\lambda) \sum_{B \subset \Delta_+(I,b)} (-1)^{|\Delta_+ (I,b)| - |B|} \sum_{A \subset \Delta(\mathfrak{g}[1], b)} (-1)^{|A|} [\text{gr } l(\lambda + 2\rho_L - 2\rho(A) - 2\rho(B))].
\]

Rewrite with \([\text{gr } l(\lambda')]\), \(\lambda'\) dominant.

Loop: find largest \(\lambda'\).

If \(\lambda' \leftrightarrow (\xi', \tau')\) for smaller \(G \cdot \xi'\), subtract

\(m_{\sigma(\tau)}(\lambda') \times \text{formula for } (\xi', \tau')\);

\(\leftrightarrow\) new formula for \((\xi, \tau)\) with smaller leading term.

When loop ends, \(\lambda' = \lambda(\xi, \tau)\).
What to do next

Sketched effective algorithms for computing assoc cycles for HC modules, Lusztig bijection.
What should we (this means you) do now?
Software implementations of these?

- Pramod Achar \rightsquigarrow \texttt{gap} script for Lusztig bij in type A.
- Marc van Leeuwen \rightsquigarrow \texttt{atlas} software for (std rep)|_K.

Real group version of Lusztig bijection?
Algorithm still works, but bijection not quite true.
Failure partitions \( \hat{K} \) into small finite sets.

Closed form information about algorithms?
formula for smallest \( \lambda \leftrightarrow \) (one orbit, any \( \tau \));
Would bound below infl char of HC-mod \( \leftrightarrow \) orbit.