

## Invariant measures on homogeneous spaces

Since I screwed up a discussion of integration formulas in class on Monday, I have prepared these notes with some of the basic facts. Missing proofs may be found for example in Leopold Nachbin's book *The Haar Integral*.

So suppose  $G$  is a locally compact topological group. I will write  $C_c(X)$  for the space of compactly supported continuous (complex-valued) functions on a locally compact topological space  $X$ , and  $C_c^{\geq 0}(X)$  for the collection of non-negative real-valued functions. I am interested in Borel measures on  $X$ ; equivalently (Riesz representation theorem) in linear functionals on  $C_c(X)$  that take non-negative values on  $C_c^{\geq 0}(X)$ .

If  $G$  acts on  $X$ , then it acts on functions on  $X$  by the formula

$$(\lambda(g)f)(x) = f(g^{-1} \cdot x). \tag{1a}$$

I'll use this notation in particular for the *left* action of  $G$  on itself. For the *right* action of  $G$  on itself, I'll write

$$(\rho(g)f)(x) = f(xg). \tag{1b}$$

The basic theorem about Haar measure is

**Theorem 2.** *Suppose  $G$  is a locally compact topological group. Then there is a non-zero Borel measure  $d_G^\ell$  on  $G$  with the property ("left invariance") that*

$$\int_G f(g^{-1}x) d_G^\ell(x) = \int_G f(x) d_G^\ell(x) \quad (g \in G, f \in C_c(G)). \tag{2a}$$

*This property characterizes  $d_G^\ell$  up to a positive scalar multiple. Any right translate of  $d_G^\ell$  shares this left-invariance, and therefore is a positive scalar multiple of  $d_G^\ell$ :*

$$\int_G f(xg) d_G^\ell(x) = \delta_G(g^{-1}) \int_G f(x) d_G^\ell(x) \quad (h \in G, f \in C_c(G)). \tag{2b}$$

*Here  $\delta_G: G \rightarrow \mathbb{R}^+$  is a continuous homomorphism, called the "modular character" of  $G$ .*

Because any compact subgroup of the multiplicative group  $\mathbb{R}^+$  is trivial, the modular character must be trivial on any compact subgroup of  $G$ .

There is a right Haar measure on  $G$  defined by

$$\int_G f(x) d_G^r(x) = \int_G f(x) \delta_G(x^{-1}) d_G^\ell(x) \quad (f \in C_c(G)). \tag{3a}$$

(The right-invariance of this linear functional on  $C_c(G)$  is an easy consequence of (2b).)

If  $G$  is a Lie group, then an elementary calculation shows that

$$\delta_G(g) = |\det(\text{Ad}(g))|. \tag{3b}$$

This formula makes it easy to construct examples of groups with non-trivial modular function (like the group of upper-triangular invertible matrices).

Suppose now that  $H$  is a closed subgroup of  $G$  (still assumed locally compact). Fix (left) Haar measures  $d_G^\ell$  on  $G$  and  $d_H^\ell$  on  $H$ . One nice way to relate integration on  $G$ ,  $H$ , and  $G/H$  is using

**Theorem 4** (Weil?). *The mapping*

$$C_c(G) \rightarrow C_c(G/H), \quad f \mapsto \bar{f}, \quad \bar{f}(xH) = \int_H f(xh) d_H^\ell(h) \quad (x \in G) \quad (4a)$$

is a well-defined surjection, carrying non-negative functions to non-negative functions.

What we would like to do is use this description of functions on  $G/H$  to introduce an integral on  $G/H$  by requiring

$$\int_{G/H} \bar{f}(xH) d_{G/H}(xH) \stackrel{?}{=} \int_G f(x) d_G^\ell(x) \quad (f \in C_c(G)). \quad (5a)$$

In order to do this (using Theorem 4 and the Riesz representation theorem) we need to know that the integral on the right must vanish whenever  $\bar{f} = 0$ .

Here is where matters become a little subtle. Suppose that  $f \in C_c(G)$  gives rise by Theorem 4 to the function  $\bar{f} \in C_c(G/H)$ . For each element  $h_0 \in H$ , the right translate  $\rho(h_0)f$  is another function in  $C_c(G)$ , and it seems only fair that it should give rise to the same function  $\bar{f}$  on  $G/H$ . But calculation shows that

$$\overline{\rho(h_0)f} = \delta_H(h_0^{-1})\bar{f}. \quad (5b)$$

Well, that's not bad: we can write it as

$$\overline{\delta_H(h_0)\rho(h_0)f} = \bar{f}. \quad (5c)$$

That is, all functions of the form

$$f - \delta_H(h_0)[\rho(h_0)f] \quad (f \in C_c(G), h_0 \in H) \quad (5d)$$

belong to the kernel of the “bar” map to  $C_c(G/H)$ . It's not difficult to prove that their span is dense in the kernel.

On the other hand, we calculate (from (2b))

$$\int_G (\rho(h_0)f)(x) d_G^\ell(x) = \delta_G(h_0^{-1}) \int_G f(x) d_G^\ell(x) \quad (f \in C_c(G), h_0 \in H). \quad (5e)$$

That is, functions of the form

$$f - \delta_G(h_0)[\rho(h_0)f] \quad (f \in C_c(G), h_0 \in H) \quad (5f)$$

have integral zero over  $G$ .

Comparing (5d) and (5f), we see that the integral in (5a) cannot vanish on all functions with  $\bar{f} = 0$  unless the modular function  $\delta_H$  is equal to the restriction of the modular function  $\delta_G$  to  $H$ . In light of the claim that the functions in (5d) span a dense subspace of the kernel, the converse holds as well. Summarizing,

**Theorem 6.** *Suppose  $G$  is a locally compact group,  $H$  a closed subgroup, and  $d_G^\ell$  and  $d_H^\ell$  are left Haar measures.*

- a) *There is a non-zero  $G$ -invariant Borel measure on  $G/H$  if and only if the modular functions of  $G$  and  $H$  satisfy  $\delta_H = (\delta_G)|_H$ .*

b) Suppose that the modular functions agree. Then the measure is unique up to a positive scalar multiple. It may be normalized so that

$$\int_G f(x) d_G^\ell(x) = \int_{G/H} \left[ \int_H f(xh) d_H^\ell(h) \right] d_{G/H}(xH) \quad (f \in C_c(G)).$$

Having come so far, I should say a few more words. If  $G$  is a Lie group, then  $G/H$  is a smooth manifold. You may know that any smooth manifold  $M$  admits a real line bundle called the “density bundle,” whose sections are the smooth measures on  $M$  (and can therefore be integrated). Having an invariant measure on  $G/H$  means that the density bundle is equivariantly trivial.

Now line bundles on  $G/H$  correspond to characters of the subgroup  $H$ ; sections of a line bundle may be identified with functions on  $G$  transforming by the character under right translation by  $H$ . The conclusion from all this (for Lie groups) is that densities on  $G/H$  correspond to functions on  $G$  transforming on the right by a certain character of  $H$ , and that for such functions there is a natural “integral” over  $G/H$ . This is simple enough that something similar might be true for general locally compact groups, and this turns out to be the case.

**Definition 7.** Suppose  $G$  is a locally compact group,  $H$  a closed subgroup, and  $d_G^\ell$  and  $d_H^\ell$  are left Haar measures. A *compactly supported continuous density* on  $G/H$  is a continuous function  $f$  on  $G$  such that

- a)  $f(xh) = \delta_H(h)\delta_G(h^{-1})f(x)$  for all  $x \in G$  and  $h \in H$ , and
- b) there is a compact subset  $K$  of  $G$  such that  $f$  is supported in  $KH$ .

**Theorem 8.** In the setting of Definition 7, the mapping

$$f \rightarrow \tilde{f}, \quad \tilde{f}(x) = \int_H f(xh)\delta_G(h)\delta_H(h^{-1})d_H^\ell(h)$$

is a continuous surjection from  $C_c(G)$  to compactly supported continuous densities on  $G/H$ . There is a left-invariant “integral” for such densities given by

$$\int_{G/H} \tilde{f} = \int_G f(x) d_G^\ell(x) \quad (f \in C_c(G)).$$

That is, the left-invariant integral over  $G$  may be expressed as an iterated integral

$$\int_G f(x) d_G^\ell(x) = \int_{G/H} \left[ \int_H f(xh)\delta_G(h)\delta_H(h^{-1})d_H^\ell(h) \right] d_{G/H}(xH) \quad (f \in C_c(G)).$$

Here the inner integral represents a compactly supported density on  $G/H$ .

Just as for Theorem 6, the key is to understand the kernel of the map “tilde”; this time it turns out to be generated by elements  $f - \delta_G(h_0)[\rho(h_0)f]$ .

The integrals over  $H$  can be written with respect to right Haar measure  $d_H^r(h)$ , by dropping the factors  $\delta_H(h^{-1})$  (cf. (3a)).