Branching to maximal compact subgroups

David Vogan

Department of Mathematics
Massachusetts Institute of Technology

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Outline

Introduction

Helgason’s theorem classically

Helgason’s theorem and algebraic geometry

Interpreting the branching law: Zuckerman’s theorem

Relating representations of $K$ and $G$
Why restrict to $K$?

$G$ cplx $\supset G(\mathbb{R})$ real $\supset K(\mathbb{R})$ maxl compact

Want to study representations $(\pi, \mathcal{H}_\pi)$ of $G(\mathbb{R})$, but these are complicated and difficult.

Reps of $K(\mathbb{R})$ are easy, so try two things:

- understand $\pi|_{K(\mathbb{R})}$; and
- use understanding to answer questions about $\pi$.

Sample question: how often does trivial representation of $K(\mathbb{R})$ appear in $\pi|_{K(\mathbb{R})}$? Answer: multiplicity zero unless $\pi$ is (quotient of) spherical principal series, then one.

Application: $\pi$ can appear in functions on $G(\mathbb{R})/K(\mathbb{R})$ only if $\pi$ spherical; then exactly once.
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Write $\theta = \text{Cartan involution of } G(\mathbb{R}) \text{ and } G$;

\[ K(\mathbb{R}) = G(\mathbb{R})^\theta \quad \text{(real groups)}, \]
\[ K = G^\theta \quad \text{(complex algebraic groups)}. \]

Iwasawa decomposition $G(\mathbb{R}) = K(\mathbb{R})A(\mathbb{R})_0N(\mathbb{R})$.

Here $A = \text{maxl cplx torus where } \theta \text{ acts by inverse}$.

$L(\mathbb{R}) = \text{centralizer of } A \text{ in } G(\mathbb{R})$

$P(\mathbb{R}) = L(\mathbb{R})N(\mathbb{R})$.

Group $P(\mathbb{R})$ is \textit{minimal parabolic subgroup of } $G(\mathbb{R})$. 
Minimal parabolic subgroup

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Helgason’s theorem (classical picture)

Minimal parabolic is \( P(\mathbb{R}) = M(\mathbb{R})A(\mathbb{R})_0N(\mathbb{R}) \).

Fin-diml of \( G(\mathbb{R}) \) \( \Leftrightarrow \) highest wt = \( N(\mathbb{R}) \)-invts.

highest weight = \( \delta \otimes \nu \), \( \delta \in \hat{M}(\mathbb{R}) \), \( \nu \in \hat{A}(\mathbb{R})_0 \).

Theorem (Helgason)

1. Rep of hwt vector has \( K(\mathbb{R}) \)-fixed vecs \( \iff \delta \) \( \text{triv.} \)
2. \( \text{triv} \otimes \nu \) is a highest wt \( \iff \nu \) is dom even int wt.

Says: \( K(\mathbb{R}) \)-fixed vecs \( \leftrightarrow \) \( M(\mathbb{R})N(\mathbb{R}) \)-fixed vecs.

Reason: \( M(\mathbb{R})N(\mathbb{R}) = \text{deformation of } K(\mathbb{R}) \).

Conjugate \( K(\mathbb{R}) \) by elts of \( A(\mathbb{R})_0 \), \( \rightsquigarrow \) limiting subgroup.
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Theorem (Helgason)

1. Rep of hwt $\delta \otimes \nu$ has $K(\mathbb{R})$-fixed vecs $\iff$ $\delta$ triv.
2. $\nu$ is a highest wt $\iff$ $\nu$ is dom even int wt.

Says: $K(\mathbb{R})$-fixed vecs $\leftrightarrow$ $M(\mathbb{R})N(\mathbb{R})$-fixed vecs.

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Theorem (Helgason)

1. Rep of highest wt has $K(\mathbb{R})$-fixed vct $\iff$ triv.
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Says: $K(\mathbb{R})$-fixed vecs $\leftrightarrow$ $M(\mathbb{R})N(\mathbb{R})$-fixed vecs.
Reason: $M(\mathbb{R})N(\mathbb{R})$ = deformation of $K(\mathbb{R})$.

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Moving toward algebraic geometry

Theme: complexify, use algebraic geometry.

Helgason’s theorem concerns compact $K(\mathbb{R})$, minimal parabolic $P(\mathbb{R})$. Theme says complexify, considering algebraic groups $K = G^\theta$ and $P = LN$ parabolic in $G$. Continuous reps of $K(\mathbb{R}) \hookrightarrow$ algebraic reps of $K$. Theme says consider projective algebraic variety $\mathcal{P} = \text{subgps of } G \text{ conjugate to } P$, a partial flag variety.
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Helgason’s theorem (alg geometry picture)

Proposition

\[ K \cdot P \text{ is open in } \mathcal{P} : K/M \cong K \cdot P \subset \mathcal{P} \cong G/P. \]  
Here \( M = \text{cplx pts of } M(\mathbb{R}) = \text{cent in } K \text{ of } A. \)

Follows immediately from Iwasawa decomposition.

Theorem (Borel-Weil, Helgason)

1. Alg repn of \( G \) = alg secs of equiv vector bdle on \( P \).
2. Gives \( \{ \text{ irr alg reps of } L \} \) \( \hookrightarrow \) \( \{ \text{ irr alg reps of } G \} \).
3. \( \hat{L} = \{ (\delta, \nu) \in \hat{M} \times \hat{A} | \delta = \nu \text{ on } M \cap A \} \).
4. \( \{ \text{ alg secs of } \delta \otimes \nu \text{ on } P \} \hookrightarrow \) \( \{ \text{ alg secs of } \delta \text{ on } K/M \} \).
5. \( \{ \text{ Alg rep of } G \mid K \hookrightarrow \) Ind \( K/M \{ \text{highest wt} \} \text{ on } M \} \).

Picture: \( \mathcal{P} = K/M \cup \{ \text{divisors} \} \).

Section on \( K/M \) \( \leadsto \) pole order on each divisor.
Section extends to \( \mathcal{P} \) \( \iff \) no pole on any divisor.
Helgason’s theorem (alg geometry picture)

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\( K \cdot P \) is open in \( \mathcal{P} : K/M \cong K \cdot P \subset \mathcal{P} \cong G/P. \) Here \( M = \text{cplx pts of } M(\mathbb{R}) = \text{cent in } K \text{ of } A. \)

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Theorem (Borel-Weil, Helgason)

1. Alg repn of \( G = \text{alg secs of equiv vector bdle on } \mathcal{P}. \)
2. Gives \( \{ \text{irr alg reps of } G \} \rightarrow \{ \text{irr alg reps of } L \}. \)
3. \( \hat{L} = \{ (\delta, \nu) \in \hat{M} \times \hat{A} : \delta = \nu \text{ on } M \cap A \}. \)
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5. \( \{ \text{Alg rep of } G \} |_{K} \rightarrow \text{Ind } K/M(\{ \text{highest wt} \} |_{M}). \)

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**Theorem (Borel-Weil, Helgason)**

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5. \( \{ \text{Alg rep of } G \} |_{K} \hookrightarrow \text{Ind}_{K/M} \{ \text{highest wt} \} |_{M} \).

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Section on \( K/M \Rightarrow \text{pole order on each divisor.} \)

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Proposition

\( K \cdot P \) is open in \( \mathcal{P} \): \( K/M \cong K \cdot P \subset \mathcal{P} \cong G/P \). Here \( M = \text{cplx pts of } M(\mathbb{R}) = \text{cent in } K \text{ of } A \).

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2. Gives \( \{ \text{irr alg reps of } G \} \hookrightarrow \{ \text{irr alg reps of } L \} \).
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Picture: \( \mathcal{P} = K/M \cup \{ \text{divisors} \} \).

Section on \( K/M \rightsquigarrow \text{pole order on each divisor.} \)

Section extends to \( \mathcal{P} \iff \text{no pole on any divisor.} \)
Helgason’s theorem (alg geometry picture)

Proposition

\[ K \cdot P \text{ is open in } \mathcal{P} : K/M \cong K \cdot P \subset \mathcal{P} \cong G/P. \] 
Here \( M = \text{cplx pts of } M(\mathbb{R}) = \text{cent in } K \text{ of } A. \)

Follows immediately from Iwasawa decomposition.

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Section on \( K/M \leadsto \) pole order on each divisor.
Section extends to \( \mathcal{P} \Longleftrightarrow \) no pole on any divisor.
Proposition

\( K \cdot P \) is open in \( \mathcal{P} \): \( K/M \simeq K \cdot P \subset \mathcal{P} \simeq G/P \). Here \( M = \text{cplx pts of } M(\mathbb{R}) = \text{cent in } K \text{ of } A \).

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**Theorem (Borel-Weil, Helgason)**

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Section on \( K/M \) \( \sim \) pole order on each divisor.

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\[ \mathcal{P} \]

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Section on \( K/M \sim \) pole order on each divisor.

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Geometric branching law from $G$ to $K$

Repn of $G =$ alg secs of vector bdle on $\mathcal{P}$

$\mathcal{P} = K/M \cup \{ \text{divisors } D_1, \ldots, D_r \}$

Divisors correspond to simple restricted roots of $A$.

Bdle on $\mathcal{P} \rightsquigarrow l_0(\delta) = \text{Ind}_M^K(\delta) =$ secs on $K/M$.

Bdle on $\mathcal{P} \rightsquigarrow l_j(\tau_j) =$ secs with pole along divisor $D_j$.

Bundle $\tau_j$ depends on $\delta$ and on character $\nu$ of $A$.

If $\nu$ large on simple root $j$, then $l_j(\tau_j)$ is small.

Sections on $\mathcal{P} \approx l_0(\delta) - \sum_{j=1}^r l_j(\tau_j)$.

Branching law: describes restr to $K$ of rep of $G$.

As $\nu$ tends to infinity, $G$ representation grows toward $l_0(\delta)$. 
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Terms in the branching law

\[(\text{fin diml rep of } G)|_K \approx l_0(\delta) - \sum_{j=1}^{r} l_j(\tau_j).\]

What do the terms on the right mean?

Classical picture:

\[l_0(\delta) = \text{Ind}_{K}^{G} (\mathbb{R}) (\delta) = \left( \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} (\delta \otimes \nu \otimes 1) \right)|_{K(\mathbb{R})},\]

restr to \(K(\mathbb{R})\) of principal series rep of \(G(\mathbb{R})\).

\(l_0\) = inf-diml rep \(l_0\), containing \(F\) as a subrep.

Geometry: \(G(\mathbb{R})/P(\mathbb{R}) = P(\mathbb{R})\) is nice real subvariety of \(G/P = P\).

\[l_0 = \text{analytic sections of bundle on } P(\mathbb{R})\]

\(F = \text{sections extending holomorphically to } P\).

Later \(l_j\) are \(G(\mathbb{R})\) reps \(\leftrightarrow\) other pieces of \(l_0\).
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Branching to maximal compact subgroups

David Vogan

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Later \( l_j \) are \( G(\mathbb{R}) \) reps \( \lhd \) other pieces of \( l_0 \).
Terms in the branching law

\[(\text{fin diml rep of } G)|_K \approx l_0(\delta) - \sum_{j=1}^{r} l_j(\tau_j).\]

What do the terms on the right mean?

**Classical picture:**

\[l_0(\delta) = \text{Ind}^{K(\mathbb{R})}_{M(\mathbb{R})}(\delta) = \left(\text{Ind}^{G(\mathbb{R})}_{P(\mathbb{R})}(\delta \otimes \nu \otimes 1)\right)|_K(\mathbb{R}),\]

restr to \(K(\mathbb{R})\) of principal series rep of \(G(\mathbb{R})\).

\(l_0 = \text{inf-diml rep } l_0\), containing \(F\) as a subrep.

Geometry: \(G(\mathbb{R})/P(\mathbb{R}) = \mathcal{P}(\mathbb{R})\) is nice real subvariety of \(G/P = \mathcal{P}\).

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Setting for Zuckerman’s theorem

First replace $\mathcal{P}$ by $B = \text{Borel subgroups of } G$, complete flag variety for $G$.

Proposition (Wolf, Beilinson-Bernstein)

1. $K$ acts on $B$ with finitely many orbits.
2. Unique open orbit $\leftrightarrow$ Borel subgp of Iwasawa $P$.
3. General orbit $\leftrightarrow$ pair $(H, \Delta^+)$ mod $G(\mathbb{R})$ conjugation.
   - $H(\mathbb{R}) = \text{Cartan in } G(\mathbb{R})$, $\Delta^+ = \text{pos roots for } H$ in $G$.
4. Std rep $I(\tau)$ restr to $K$ on $\xi_{\tau}$ ($\mathbb{R}$).
   - $\xi = \text{orbit of } \xi$ on $\xi_{\tau}$, $\xi$ equiv line bundle on $Z$. 
   - Bundle $\xi \leftrightarrow \text{alg char of } H \cap K$.

Bundle must be “positive” (as in Borel-Weil).
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   \[ H(\mathbb{R}) = \text{Cartan in } G(\mathbb{R}), \quad \Delta^+ = \text{pos roots for } H \text{ in } G. \]
4. Std rep $l(\tau)$ restr to $K \sim (Z, \xi)$ 
   
   \[ Z = \text{orbit of } K \text{ on } \mathcal{B}, \xi \text{K-eqvt line bdle on } Z. \]

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$F$ finite-diml irr rep of $G \leadsto$ line bdle $\lambda$ on $B$.

$K$-orbit $Z \subset B \leadsto$ parameter $\tau(Z, F) = (Z, \lambda|Z)$.

For each $K$-orbit $Z$, std rep restr to $K I(\tau(Z, F))$.

Theorem (Zuckerman)

If $F$ any fin-diml irr rep of $G$ (cplx reductive), then

$$F|_K = \sum_{Z \subset B} (-1)^{\text{codim}(Z)} I(\tau(Z, F)).$$

Sum is over orbits of $K$ (complexified max compact) on flag variety $B$.

1st term (codim 0) $\leadsto$ princ series $\leftrightarrow M$ rep $F^N$.

Next terms (codim 1) $\leadsto$ poles on divisors $\mathcal{P} - K/M$.

Higher terms correct for double counting.
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Example: $\text{SL}(2, \mathbb{C})$

\[ G(\mathbb{R}) = \text{SL}(2, \mathbb{C}), \ K(\mathbb{R}) = \text{SU}(2). \]

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Finite diml of $G \rightsquigarrow F_{a,b} = \mathbb{C}^a \otimes \mathbb{C}^b$, ($a$ and $b$ pos ints).

Irr of $K \rightsquigarrow \tau_x =$ highest weight $x$ ($x \in \mathbb{N}$).

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F_{a,b}|_K = I(a - b) - I(a + b) \\
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Helgason thm when fin diml restr to $K$ is spherical led us to Zuckerman thm descr of fin diml restr to $K$. What’s the next step?

Zuckerman formula (fin diml) $= (\text{alt sum of std reps})$ suggests (any irr rep) $? (\text{integer comb of std reps}).$

Leads to Kazhdan-Lusztig theory, not dull.

But orig Helgason thm suggests instead looking for formulas (irr of $K$) $? (\text{alt sum of std reps}).$

Application: invert the matrix above to get branching laws (std rep for $G(\mathbb{R})$) $= (\text{sum of irrs of } K)$.

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Won’t write theorem for general $G$ (painful notation); pass directly to examples...
**Example:** \( SL(2, \mathbb{C}) \)

\[
G(\mathbb{R}) = SL(2, \mathbb{C}), \quad K(\mathbb{R}) = SU(2).
\]

Irr of \( K \) \( \leadsto \) \( \tau_x \) = rep of highest wt \( x \) \( (x \in \mathbb{N}) \).

Std rep \( l(m) \) = sum of reps of \( K \) cont. wt \( m \) \( (m \in \mathbb{Z}) \).

Write each irr of \( K \) = alt sum of std reps of \( G(\mathbb{R}) \).

\( m + 1 \)-diml irr of \( K \) is \( \tau_m = l(m) - l(m + 2) \).

Invert:

\[
l(m) = (l(m) - l(m + 2)) + (l(m + 2) - l(m + 4)) + (l(m + 4) - l(m + 6)) \cdots
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= \tau_m + \tau_{m+2} + \tau_{m+4} \cdots.
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\[ G(\mathbb{R}) = SL(2, \mathbb{R}), \quad K(\mathbb{R}) = SO(2). \]

Chars of \( K \) \( \leadsto \) \( \tau_k \quad (k \in \mathbb{Z}). \)

Princ series \( I^{ps} = \) sph princ series restr to \( K \).

Hol disc series \( I^+ (m) \quad (m \in \mathbb{N} \text{ HC param}). \)

Antihol disc series \( I^- (m) \quad (m \in -\mathbb{N} \text{ HC param}). \)

Write each irr of \( K = \) alt sum of std reps of \( G(\mathbb{R}). \)

\[ \tau_0 = I^{ps} - I^+ (0) - I^- (0) \]

\[ \tau_m = I^+ (m - 1) - I^+ (m + 1) \quad (m > 0). \]

\[ \tau_m = I^- (m + 1) - I^- (m - 1) \quad (m < 0). \]

Invert:

\[ I^{ps} = (I^{ps} - I^+ (0) - I^- (0)) + (I^+ (0) - I^+ (2)) + (I^- (0) - I^- (-2)) + \cdots \]

\[ = \tau_0 + \tau_2 + \tau_{-2} + \cdots \]

\[ I^+ (m) = (I^+ (m) - I^+ (m+2)) + (I^+ (m+2) - I^+ (m+4)) + \cdots \]

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Invert:

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$= \tau_0 + \tau_2 + \tau_{-2} + \cdots$

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I^{ps} &= (I^{ps} - I^+(0) - I^-(0)) + (I^+(0) - I^+(2)) + (I^-(0) - I^-(2)) + \cdots \\
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\newline

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Summary

- **Helgason’s theorem** on spherical fin-diml reps connects Borel-Weil picture of fin-diml reps. to inf-diml reps.
- Zuckerman’s theorem extends this to description of fin-diml rep as alt sum of “standard” inf-diml reps.
- Variation on this theme writes any fin-diml of $K$ as alt sum of standard inf-diml reps.
- Inverting these formulas writes standard inf-diml as sum of irrs of $K$. 
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