

# THE CONJUGACY CLASSES OF $GL(2, \mathbb{F}_q)$

HAROLD COOPER

## 1. BACKGROUND

**Definition 1.** Recall that the *conjugacy class* of some  $x \in G$  is  $C_x = \{g x g^{-1} | g \in G\}$  and that the *stabilizer* of  $x$  is  $Z(x) = \{g \in G | g x g^{-1} = x\}$ .

Recall furthermore the counting formula, for any  $x \in G$ :

$$(1.1) \quad |G| = |C_x| \cdot |Z(x)|$$

Lastly, it was previously presented that the order of  $GL(2, \mathbb{F}_q)$  is:

$$(1.2) \quad |GL(2, \mathbb{F}_q)| = (q+1)q(q-1)^2$$

## 2. THE CLASSES

For convenience we let  $G = GL(2, \mathbb{F}_q)$ .

We proceed to consider the conjugacy classes of  $G$ .

- (1) First let us consider matrices which can be transformed by conjugation into the form  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ , where  $a \neq b$ .

In other words, we are considering diagonalizable matrices with distinct eigenvalues. This gives a distinct conjugacy class for each set of distinct eigenvalues  $\{a, b\}$ , since diagonalizability and eigenvalues are preserved by conjugation.

For a given  $a$  and  $b$  we will let  $g_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$  be the representative element of its conjugacy class. Note that the stabilizer of  $g_{a,b}$  is the set of all diagonal matrices in  $G$ , of which there are  $(q-1)^2$ .

So by our counting formula, each of the conjugacy classes we are considering has size:

$$|C_{g_{a,b}}| = |G|/|Z(g_{a,b})| = (q+1)q(q-1)^2/(q-1)^2 = (q+1)q$$

There are  $\binom{q-1}{2} = \frac{(q-1)(q-2)}{2}$  choices of  $a$  and  $b$ , each of which yields a distinct conjugacy class of size  $(q+1)q$ , so these conjugacy classes contribute

$$\frac{(q+1)q(q-1)(q-2)}{2}$$

elements to  $G$ .

- (2) Next we consider diagonalizable matrices with just *one* eigenvalue. These are simply the matrices of the form  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ , since such matrices are fixed by conjugation, so they are the only diagonalizations of themselves.

Indeed, the fact that such matrices are fixed by every conjugation means that their stabilizers consist of all of  $G$ , so for a given  $g_a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ , we have:

$$|C_{g_a}| = |G|/|Z(g_a)| = |G|/|G| = 1$$

That is to say, each conjugacy class of this type has just 1 element. And since there are  $q - 1$  such conjugacy classes (one for each nonzero  $a$ ), these contribute

$$q - 1$$

elements to  $G$ .

- (3) Now we consider elements  $x \in G$  which are *not* diagonalizable, but still have eigenvalues in  $\mathbb{F}_q$ .

Note that this implies that the eigenvalues of such  $x$  are equal, since if they were not, conjugating the diagonal matrix consisting of the eigenvalues of  $x$  by the matrix made up of the eigenvectors would yield  $x$  itself, i.e.  $x$  would be diagonalizable.

Since  $x$  is conjugate to any other matrix which has the same eigenvalues and is also not diagonalizable, we can simply focus on representative matrices of the form

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

The stabilizers of such matrices are of the form  $\begin{pmatrix} b & c \\ 0 & b \end{pmatrix}$  for any  $b, c$  where  $b \neq 0$ .

Thus the stabilizers have size  $q(q - 1)$ , so the conjugacy classes have size:

$$|C| = |G|/|Z| = (q + 1)q(q - 1)^2/q(q - 1) = (q + 1)(q - 1)$$

There are  $q - 1$  such conjugacy classes, one for each nonzero  $a$  in  $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ , so this type of conjugacy class contributes

$$(q + 1)(q - 1)^2$$

elements to  $G$ .

- (4) Finally, we consider the only remaining elements of  $G$ : matrices with eigenvalues  $\lambda_1, \lambda_2$  in the quadratic extension field  $\mathbb{F}_{q^2}$  of  $\mathbb{F}_q$ .

By definition both  $\lambda_1$  and  $\lambda_2$  are roots of the same characteristic quadratic polynomial.

But quadratic polynomials have at most two roots, and  $\lambda_1$  and its 'conjugate',  $\phi(\lambda_1)$ , are two such roots (where  $\phi$  is the canonical automorphism on  $\mathbb{F}_{q^2}$  which fixes  $\mathbb{F}_q$ ) so it must be the case that  $\lambda_2 = \phi(\lambda_1)$ .

Thus there is a different conjugacy class for each one of the  $q^2 - q = q(q - 1)$  possible values of  $\lambda_1$ , but the order of the eigenvalues does not matter, so we have  $\frac{q(q-1)}{2}$  different conjugacy classes of this type.

It is difficult to reason about the stabilizers of such matrices this late at night, but fortunately this is the last set of conjugacy classes, and we already know the size of  $G$ , and that conjugacy classes partition  $G$ , so we can simply subtract to determine the number of elements which this last batch of classes should contribute.

This yields a required contribution of

$$\frac{q^2(q-1)^2}{2}$$

elements to  $G$ , from which we can deduce that the size of each conjugacy class is  $q(q-1)$  and thus the size of the stabilizers is

$$|Z| = |G|/|C| = (q+1)q(q-1)^2/q(q-1) = (q+1)(q-1)$$

which of course we could've figured out in plenty of other ways, but is certainly correct nonetheless.

Thus we have found and described *every* conjugacy class of  $G = GL(2, \mathbb{F}_q)$ !

If we hadn't already used the fact that the conjugacy classes ought to partition  $G$ , we could at this point add up all of their sizes to double-check our computations. Alas, that would be rather meaningless now.