Errata for “Parameters for twisted representations”

Jeffrey D. Adams∗
Department of Mathematics
University of Maryland

David A. Vogan, Jr.†
2-355, Department of Mathematics
MIT, Cambridge, MA 02139

June 29, 2017

1 Introduction

The article [1] describes an algorithm for computing the unitary dual of a real reductive algebraic group $G(\mathbb{R})$. One ingredient in the algorithm is the Kazhdan-Lusztig polynomials defined and computed in [4]. These polynomials are indexed by pairs $(J, J')$ of irreducible representations of $G(\mathbb{R})$.

A second ingredient in the unitarity algorithm is a twisted version of these polynomials introduced in [5]. The setting involves an outer automorphism $\delta$ of $G(\mathbb{R})$ of order two, and the corresponding extended group $\delta G(\mathbb{R})$ (containing $G(\mathbb{R})$ as a subgroup of index two). These twisted polynomials are indexed by pairs $(\tilde{J}, \tilde{J}')$ of extensions to $\delta G(\mathbb{R})$ of irreducible representations of $G(\mathbb{R})$. Each $\delta$-fixed irreducible $J$ of $G(\mathbb{R})$ admits exactly two extensions $\tilde{J}_{\pm 1}$ and $\tilde{J}_{\mp 1}$ to $\delta G(\mathbb{R})$. Roughly speaking, the twisted polynomials depend only on the underlying $G(\mathbb{R})$ representations. Precisely, if $\tilde{J}_{\pm 1}$ are the two extensions of a $G(\mathbb{R})$ irreducible $J$, and $\tilde{J}'_{\pm 1}$ the two extensions of $J'$, then

$$P_{J,J'} = \epsilon \phi P_{\tilde{J}_{\pm 1}, \tilde{J}'_{\pm 1}}.$$ 

The difficulty is that (despite the misleading notation $\tilde{J}_{\pm 1}$) there is no preferred extension of $J$ to $\delta G(\mathbb{R})$. A representation like $J$ can be specified precisely using (any of various versions of) a Langlands parameter $p$. The point of the paper [2] was to introduce extended parameters $E$ ([2, Definition 5.4]). An extended parameter consists of a Langlands parameter $p$ and some additional
data (for which there are up to equivalence exactly two choices). The Langlands parameter specifies an irreducible $J(p)$ for $G(\mathbb{R})$. The equivalence class of $E$ specifies precisely one extension $J(E)$ to $\delta G(\mathbb{R})$.

Given this precise specification of extended group representations, the algorithm of [5] could be formulated in terms of extended parameters $E$. This formulation was also presented in [2], and it is there that (at least one) error arose.

Here is the nature of the error. The algorithms of [5] involve various linear maps $T_\kappa$ defined on $\mathbb{Z}_q$-linear combinations of extended group representations. These formal linear combinations are subject to the relations

\[ \tilde{J}_{i+1} = -\tilde{J}_{i-1}. \]

A typical step in the algorithm involves two to four representations $J_i$ and says something like this: extensions $\tilde{J}_i$ of $J_i$ may be chosen so that

\[ T_\kappa(\tilde{J}_1) = \tilde{J}_1 + \tilde{J}_3 + \tilde{J}_4, \quad T_\kappa(\tilde{J}_2) = \tilde{J}_2 + \tilde{J}_3 - \tilde{J}_4 \quad (1.1) \]

(see [5, (7.6)])). If one replaces any $\tilde{J}_i$ by the other extension of $J_i$, then the sign of the coefficient of the $\tilde{J}_i$ term in each such formula must change.

For each of the cases considered in [5], there is an explanation in [2] of how to choose extended parameters so that the formulas in [5] are true. The error is that for the case 2112 described in [2, Lemma 8.1], the choices are incorrect. More precisely, the formulas [2, (44)] must be replaced by

\[ T_\kappa(E_0) = E_0 + F_0 + (-1)^{(\sigma,t)}F'_0 \]
\[ T_\kappa(E'_0) = E'_0 + F_0 + (-1)^{(\sigma,t)}F'_0 \]
\[ T_\kappa(F_0) = (q^2-1)(E_0 + E'_0) + (q^2-2)F_0 \]
\[ T_\kappa(F'_0) = (-1)^{(\sigma,t)}(q^2-1)(E_0 - E'_0) + (q^2-2)F'_0. \quad (1.2) \]

(What has been added is the factors $(-1)^{(\sigma,t)}$.) We will sketch a proof of these corrected formulas in Section 2. For the introduction, we will say a word about the source of the error. All of the formulas in [5] concern behavior of sheaves on $G$ (or rather on some version of $G$ defined over a finite field) in the direction of some very small Levi subgroup $L$ of $G$: the group $L$ is locally isomorphic to $SL(2)$, $SL(2) \times SL(2)$, or $SL(3)$, in each case times a torus factor. Standard techniques allow one to prove the formulas working in $L$ rather than in $G$; so one is ultimately making statements about the representation theory of $L(\mathbb{R})$. Standard techniques very often allow one to reduce representation-theory questions about reductive groups to the case of semisimple groups, since the center necessarily acts by scalars in an irreducible representation. This technique was used (correctly) in [5] to prove (1.1). It was used sloppily to justify [2, Lemma 8.1]. The Lemma is true when $G$ is locally isomorphic to $SL(2) \times SL(2)$; but the definitions around extended parameters allow what happens on the center to affect signs. The result is that one can construct
extended parameters for a group locally isomorphic to \( SL(2) \times SL(2) \times \mathbb{C}^\times \) for which [2, Lemma 8.1] fails.

One might hope that therefore the result is true for semisimple \( G \), but this also fails: this bad \( SL(2) \times SL(2) \times \mathbb{C}^\times \) example turns up inside \( SO(p,q) \).

Now that we have your attention, we will conclude this introduction with a much more ordinary error: the first formula
\[
\text{sgn}(E,E') = i^\langle (\delta_0-1)\lambda, t' - t \rangle + \langle \delta_0 - 1, \langle \delta, \lambda \rangle + \langle \tau, t' \rangle \rangle - \langle \tau, t' - t \rangle
\]  
\[
(1.3) \quad \text{e:6.5badsgn}
\]
from [2, Proposition 6.5] is incorrect: the plus sign between the two terms in the exponent of \( i \) should be a minus. The corrected formula is
\[
\text{sgn}(E,E') = \text{sgn}(E,E') = i^\langle (\delta_0-1)\lambda, t' - t \rangle - \langle \delta_0 - 1, \langle \delta, \lambda \rangle + \langle \tau, t' \rangle \rangle - \langle \tau, t' - t \rangle
\]
\[
(1.4) \quad \text{e:6.5goodsgn}
\]

2 Two copies of \( SL(2) \)

Here is a corrected replacement of [2, Lemma 8.2]. The hypotheses are somewhat different (roughly speaking, more general) from those of the original; after sketching a proof, we will see how this corrected statement leads to (1.2). Notation is as in [2].

**Lemma 2.1.** Suppose \( \kappa \) is of type \( 2112f \) for \( E = (\lambda, \tau, \ell, t) \). Define
\[
\ell^{\text{split}} = \ell + [(\gamma_\alpha - \ell_\alpha - 1)/2] \alpha^\vee + [(\gamma_\beta - \ell_\beta - 1)/2] \beta^\vee.
\]

Suppose that
\[
F = (\lambda', \tau', \ell^{\text{split}}, t)
\]
is an extended parameter of type \( 2r21f \) appearing in \( T_\kappa(E) \). Then the coefficient with which it appears is the ratio of the \( z \)-values for these two extended parameters (see [2, Definition 5.5]). Explicitly, this is
\[
-\frac{z(\lambda', \tau', \ell^{\text{split}}, t)}{z(\lambda, \tau, \ell, t)} = i^{\langle (\delta-1)\ell^{\text{split}} - (\tau, \delta - 1)t \rangle} (-1)^{\langle \lambda' - \lambda, t \rangle}.
\]
\[
(2.2a)
\]

**Proof.** As mentioned in the introduction, the definition of \( T_\kappa \) involves sheaves on a form of \( G \) defined over a finite field. One can make the computation entirely in the Levi subgroup of \( G \) defined by
\[
\kappa = (\alpha, \beta) = (\alpha^\vee \delta(\alpha)).
\]
\[
(2.2a)
\]
We may therefore assume that \( G \) is equal to \( L \). Writing \( Z \) for the identity component of the center of \( G \), this means that
\[
G \text{ is a quotient of } SL(2) \times SL(2) \times Z
\]
\[
(2.2b)
\]
by a finite central subgroup; the first \( SL(2) \) corresponds to \( \alpha \) and the second to \( \beta \). So there is a natural identification of Lie algebras
\[
\mathfrak{g} = \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{z}.
\]
\[
(2.2c)
\]
We use the standard torus

\[ H = \left\{ \left[ \begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right], \left[ \begin{array}{cc} y & 0 \\ 0 & y^{-1} \end{array} \right], z \right| x, y \in \mathbb{C}^\times, z \in \mathbb{Z} \right\} \]  \hspace{1cm} (2.2d)

(Note that \( H \) is a quotient of \( \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{Z} \), not a direct product.) The Lie algebra of \( H \) is identified in this way as

\[ \mathfrak{h} = \left\{ (x, y, z) \right| x, y \in \mathbb{C}^\times, z \in \mathbb{Z} \right\}. \]  \hspace{1cm} (2.2e)

The pinning is given by the simple root vectors

\[ X_\alpha = \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right], \quad X_\beta = \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]. \]  \hspace{1cm} (2.2g)

The Tits group generators are

\[ \sigma_\alpha = \left[ \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 1 \end{array} \right], \quad \sigma_\beta = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \end{array} \right]. \]  \hspace{1cm} (2.2h)

Here is the strategy of the proof. The terms \( \ell \) and \( \ell^{\text{plit}} \) in our extended parameters define strong involutions \( \xi \) and \( \xi^{\text{plit}} \), and therefore subgroups

\[ K_\xi = G^\xi, \quad K_\xi^{\text{plit}} = G^{\xi^{\text{plit}}}. \]  \hspace{1cm} (2.3a)

These have index two in the corresponding subgroups of the extended group

\[ \delta_0 K_\xi = [\delta_0 G]^\xi, \quad \delta_0 K_\xi^{\text{plit}} = [\delta_0 G]^{\xi^{\text{plit}}}. \]  \hspace{1cm} (2.3b)

The hypothesis that \( F \) appears in \( T_\nu(E) \) means in particular that \( \xi^{\text{plit}} \) is conjugate to \( \xi \) by a unique coset \( gK_\xi \).

The extended parameters \( E \) and \( F \) define

\[ J(E) = \text{irreducible } (\mathfrak{g}, \delta_0 K_\xi)-\text{module} \]
\[ I(F) = \text{standard } (\mathfrak{g}, \delta_0 K_\xi^{\text{plit}})-\text{module} \]
\[ I(F)^{\text{new}} = \text{standard } (\mathfrak{g}, \delta_0 K_\xi)-\text{module}; \]  \hspace{1cm} (2.3c)
the last is obtained by twisting $I(F)$ by $\text{Ad}(g)$.

So what is the representation-theoretic interpretation of the coefficient of $F$ in $T_\kappa(E)$? The multiplicity matrix $m$ (giving multiplicities of irreducibles $J$ as composition factors of standard modules $I$) is essentially defined by

$$I = \sum_{J \text{ irreducible}} m(J, I) J. \quad (2.3d) \quad \{e:\text{multform}\}$$

The inverse matrix $M$ writes an irreducible representation $J'$ as an integer combinations of standard representations $I'$:

$$J' = \sum_{I' \text{ standard}} M(I', J') I'. \quad (2.3e) \quad \{e:\text{charform}\}$$

That the matrices $m$ and $M$ are inverses is more or less a definition.

Suppose now that $E$ and $F$ are representation parameters differing by a single link, which is an ascent from $E$ to $F$. The entries indexed by $(E, F)$ are just one off the diagonal of these upper triangular unipotent matrices; so the inverse relationship gives

$$m(J(E), I(\pm F)_{\text{new}}) = -M(I(E), J(\pm F)_{\text{new}}). \quad (2.3f) \quad \{e:\text{linkinverse}\}$$

The Kazhdan-Lusztig polynomials actually compute dimensions of stalks of some perverse cohomology sheaves, and the character formulas (2.3e) involve those dimensions with a $(-1)^{\text{codimension}}$ factor. The conclusion is that

$$M(I(E), J(F)_{\text{new}}) - M(I(E), J(-F)_{\text{new}}) = (-1)^{l(F)-l(E)} P^{\text{tw}}_{E,F}(1). \quad (2.3g) \quad \{e:\text{KLchar}\}$$

Here $I(-F)_{\text{new}}$ means $I(F)_{\text{new}}$ tensored with the nontrivial character of $\delta_0 G/G$, the other extension of the standard representation to the extended group.

The (twisted) Kazhdan-Lusztig algorithm in our case says that

$$P^{\text{tw}}_{E,F} = \text{coeff. of } F \text{ in } T_\kappa(E). \quad (2.3h) \quad \{e:\text{Tkappachar}\}$$

Combining the last three equations gives

$$\text{coeff. of } F \text{ in } T_\kappa(E) = -(-1)^{l(F)-l(E)} \left[ m(J(E), I(F)_{\text{new}}) - m(J(E), I(-F)_{\text{new}}) \right]. \quad (2.3i) \quad \{e:\text{Tkappamult2}\}$$

In our present case of length difference 2, this is

$$\text{coeff. of } F \text{ in } T_\kappa(E) = -m(J(E), I(F)_{\text{new}}) + m(J(E), I(-F)_{\text{new}}). \quad (2.3j) \quad \{e:\text{Tkappamult2}\}$$

It turns out that exactly one of the two multiplicities on the right is nonzero, and that one is 1; so determining the sign of $F$ in $T_\kappa(E)$ means determining whether or not $J(E)$ appears in $I(F)_{\text{new}}$. If $J(E)$ does appear, the sign is $-1$; if it does not, the sign is $+1$.

Up to this point, the reduction to $SL(2) \times SL(2)$ is unimportant: we could have said the same words on the larger group $G$. But our determination of the multiplicity will use special facts about $SL(2)$. Here they are.
Lemma 2.4. Suppose we are in the setting (2.2).

1. The discrete series \((g, \delta_0 K_\xi)\)-module \(J(E)\) is uniquely determined by its infinitesimal character and (unique) lowest \(\delta_0 K_\xi\)-type.

2. If we define 
   \[
   \delta_0 K_\xi^# = \langle K_\xi^0, (\delta_0 H)^\xi \rangle = (\delta_0 H)^\xi,
   \]

   then this lowest \(\delta_0 K_\xi\)-type is
   \[
   \text{Ind}_{\delta_0 K_\xi}^{\delta_0 K_\xi^#} (\Lambda(E) \otimes \omega(\alpha, \beta)).
   \]

   Here \(\Lambda(E)\) is the character of the extended torus \((\delta_0 H)^\xi\) defined by \(E\), and \(\omega(\alpha, \beta)\) means the character by which \(\delta_0 H\) acts on the exterior algebra element \(X_\alpha \wedge X_\beta\).

3. Write \(H^{\text{new}} = \text{Ad}(g)(H)\), with \(g\) defined after (2.3b), and \(\Lambda(F^{\text{new}})\) for the corresponding one-dimensional character of \((\delta_0 H^{\text{new}})^\xi\). Then
   \[
   I(F^{\text{new}})|_{\delta_0 K_\xi} = \text{Ind}_{\delta_0 H^{\text{new}} \cap (\delta_0 K_\xi^\# \Lambda(F^{\text{new}}))}^{\delta_0 K_\xi^#} (\Lambda(F^{\text{new}})).
   \]

4. The discrete series representation \(J(E)\) is a composition factor of the principal series representation \(I(F^{\text{new}})\) if and only if
   \[
   \text{Hom}_{(\delta_0 H^{\text{new}} \cap (\delta_0 K_\xi^\# \Lambda(F^{\text{new}})))}^{\delta_0 K_\xi^#} (\Lambda(E) \otimes \omega(\alpha, \beta), \Lambda(F^{\text{new}})) \neq 0.
   \]

Proof. Part (1) is a well-known general fact about discrete series representations for reductive groups; the extension to \(\delta_0\)-fixed discrete series for extended groups is routine. Part (2) is equally general. (For general \(G\) or \(\delta_0 G\) the inducing representation is the lowest \(K_\xi^\#\)- or \(\delta_0 K_\xi^\#\)-type. The highest \((\delta_0 H)^\xi\)-weight of that representation is \(\Lambda(E)\) tensored with the top exterior power of \(n/n \cap k\).)

Part (3) is a general fact about principal series representations attached to split maximal tori.

For (4), because the infinitesimal characters of \(J(E)\) and \(I(F^{\text{new}})\) are both given by the (unwritten) parameter \(\gamma\), we just need (by (1)) to determine whether the lowest \(\delta_0 K_\xi\)-type of \(J(E)\) appears in \(I(F^{\text{new}})\). Using (2), this amounts to deciding the nonvanishing of

\[
\text{Hom}_{\delta_0 K_\xi}^{\delta_0 K_\xi^#} (\Lambda(E) \otimes (\alpha + \beta), I(F^{\text{new}})).
\]

(2.5a)

Because \(\delta_0 K_\xi^#\) meets both cosets of the inducing subgroup in (3), we get

\[
I(F^{\text{new}})|_{\delta_0 K_\xi^#} = \text{Ind}_{(\delta_0 H^{\text{new}}) \cap \delta_0 K_\xi^#}^{\delta_0 K_\xi^#} (\Lambda(F^{\text{new}}))
\]

(2.5b)
Another application of Frobenius reciprocity says that we are left with deciding the nonvanishing of

\[ \text{Hom}(\delta_0 H^\text{new}) \cap (\delta_0 H) \xi (\Lambda(E) \otimes \omega(\alpha, \beta), \Lambda(F^\text{new})) , \]  

(2.5c)
as we wished to show.

There is one dangerous point about the lemma and the notation used. The roots \( \alpha \) and \( \beta \) are well-defined characters of \( H \) and therefore of its subgroup \( H^\xi \); and \( H^\xi \) acts on \( \omega(\alpha, \beta) \) by \( \alpha + \beta \). But it is not so obvious how \( \delta_0 \) acts. As an automorphism of \( H \), \( \delta_0 \) preserves the pair of roots \( \{ \alpha, \beta \} \); so one might think that it should act trivially. But of course \( \delta_0 \) interchanges the root vectors \( X_\alpha \) and \( X_\beta \) of (2.2g), and therefore acts by \(-1\) on their exterior product:

\[ (\omega(\alpha, \beta))(\delta_0) = -1. \]  

(2.6)

In order to prove Lemma 2.1, we will write down everything explicitly, in order to compute \( (\delta_0 H^\text{new}) \cap (\delta_0 H) \xi \) and determine whether the two characters agree there.

We are concerned with multiplying \( \xi_0 \) and \( \delta_0 \) by torus elements (and, eventually, Tits group elements). This involves the map

\[ e : g \to G, \quad e(L) = \exp(2\pi i L). \]  

(2.7b)

For \( L \in \mathfrak{h} \), in the coordinates of (2.2e), this is

\[ e(L) = (\exp(\pi i L_\alpha), \exp(\pi i L_\beta), e(L_Z)). \]  

(2.7c)

If \( L \) is half-integral (so that \( 2L_\alpha \) and \( 2L_\beta \) are integers) this is

\[ e(L) = \begin{bmatrix} i^{2L_\alpha} & 0 & 0 \\ 0 & i^{-2L_\alpha} & 0 \\ 0 & 0 & i^{-2L_\beta} \end{bmatrix}, e(L_Z). \]  

(2.7d)

The strong involution of \( G \) attached to our extended parameter \( E \) is

\[ \xi = e((g - \ell)/2) \xi_0 \]

\[ = \begin{bmatrix} i^{g_\alpha - \ell_\alpha} & 0 & 0 \\ 0 & i^{g_\beta - \ell_\beta} & 0 \\ 0 & 0 & i^{(g_\alpha - \ell_\alpha) + (g_\beta - \ell_\beta) + (g_Z - \ell_Z)/2} \end{bmatrix} \xi_0. \]  

(2.7e)
Because $g_\alpha - \ell_\alpha$ and $g_\beta - \ell_\beta$ are odd (this is the "2i" part of the nature of our extended parameter) the conclusion is that

$$\xi \text{ acts on each } SL(2) \text{ factor by conjugation by } \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (2.7f)$$

In particular, the action on the standard torus $\mathbb{C}^{\times}$ is trivial; so

$$H^\xi = (\mathbb{C}^{\times}) \cdot (\mathbb{C}^{\times}) \cdot (\mathbb{Z}^{\delta_0}). \quad (2.7g)$$

(Again this fixed point group is a quotient of the direct product.) The extended parameter $E$ provides also a representative

$$\delta = e(-t/2)\delta^0 = \left[ \left( i^{-t_\alpha} 0 \\ 0 i^{t_\alpha} \right), \left( i^{-t_\beta} 0 \\ 0 i^{t_\beta} \right), e(-tZ/2) \right] \delta_0 \quad (2.7h)$$

for the other coset of $(\delta_0 H)^\xi$.

Our next task is to write down $H^{\text{new}}$. This is meant to be a pinned torus in $G$ chosen so that the strong involution $\xi(F)$, when defined with respect to the new pinned torus, is equal to $\xi$. We could write down such a pinned torus in one fell swoop, but it is perhaps a bit clearer to write down a simple choice that almost works. This is

$$H^{\text{split}} = \left\{ \left[ \begin{array}{cc} \cosh(a) & \sinh(a) \\ \sinh(a) & \cosh(a) \end{array} \right], \left[ \begin{array}{cc} \cosh(b) & \sinh(b) \\ \sinh(b) & \cosh(b) \end{array} \right], z \right\} \mid a, b \in \mathbb{C}, z \in \mathbb{Z} \right\}. \quad (2.8a)$$

The simple coroots are

$$H^{\text{split}}_\alpha = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right], 0 \right]$$

$$H^{\text{split}}_\beta = \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], 0 \right]. \quad (2.8b)$$

The pinning is given by the simple root vectors

$$X^{\text{split}}_\alpha = \left[ \begin{array}{cc} \frac{1}{2} & 1 \\ 0 & 1 \end{array} \right], \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right], 0 \right]$$

$$X^{\text{split}}_\beta = \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right], \left[ \begin{array}{cc} \frac{1}{2} & 1 \\ 1 & 1 \end{array} \right], 0 \right]. \quad (2.8c)$$

The Tits group generators are

$$\sigma^{\text{split}}_\alpha = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], 1 \right]$$

$$\sigma^{\text{split}}_\beta = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], 1 \right]. \quad (2.8d)$$

The torus $H^{\text{split}}$ with this pinning is evidently conjugate to $H$ with the original pinning by an element of $G$ of the form $(d, d, 1)$. This conjugation fixes
ξ_0 (since ξ_0 acts trivially on each SL(2) factor) and δ_0 (since δ_0 interchanges the two SL(2) factors). The distinguished involutions attached to our new Cartan and pinning are therefore unchanged:

\[ \xi_0^{\text{split}} = \xi_0, \quad \delta_0^{\text{split}} = \delta_0. \]  

(2.8e)

The equation analogous to (2.7d) says that for \( L \in \mathfrak{h} \) half-integral,

\[ e(L) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{2L_\alpha}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{2L_\beta}, e(L_Z). \]  

(2.8f) \{\text{esplit}\}

In order to compute this, it is helpful to notice that for \( m \in \mathbb{Z} \),

\[ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^m = \begin{cases} \begin{pmatrix} (-1)^{m/2} & 0 \\ 0 & (-1)^{m/2} \end{pmatrix} & \text{(m even)} \\ \begin{pmatrix} 0 & i^m \\ i^m & 0 \end{pmatrix} & \text{(m odd)} \end{cases} \]  

(2.8g)

The strong involution attached to the extended parameter \( F \) is therefore

\[ \xi^{\text{split}} = e((g - r^{\text{split}})/2)\sigma_\alpha^{\text{split}}\sigma_\beta^{\text{split}}\xi_0 \\
= \left[ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e((g_Z - \ell_Z)/2) \right] \xi_0 \]  

(2.8h) \{\text{xisplit}\}

The extended parameter \( F \) provides also a representative

\[ \delta^{\text{split}} = e(-\ell/2)\sigma_0 = \left[ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{-\ell_\alpha}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{-\ell_\beta}, e(-\ell_Z/2) \right] \delta_0 \]  

(2.8i) \{\text{deltasplit}\}

for the other coset of \((\delta_0 H)\xi\).

To get into the classical representation-theoretic picture, we need to conjugate \( \xi^{\text{split}} \) (by an element of \( H^{\text{split}} \)) to \( \xi \). The elements are written at (2.7e) and (2.8h). The key to the calculation is

\[ \text{Ad} \left( \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) \left( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \]

Writing

\[ 2a = g_\alpha - \ell_\alpha - 1, \quad 2b = g_\beta - \ell_\beta - 1 \]  

(2.8j)

(so that \( a \) and \( b \) are integers) we get

\[ \text{Ad} \left( \left[ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^a, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^b \right] \right) (\xi^{\text{split}}) = \xi. \]  

(2.8k)
Conjugating $\delta_{\text{split}}$ in the same way gives
\[
\delta_{\text{new}} = \text{Ad} \left( \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \end{bmatrix} a - b - t_{\alpha} , \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \end{bmatrix} b - a - t_{\beta} , e(-t_{Z}/2) \right) \delta_0
\]

(2.8l)

Because $(1 + \theta)t = (\delta - 1)\ell$ and $g_{\alpha} = g_{\beta}$, one finds that
\[
t_{\alpha} = -t_{\beta} = (\ell_{\beta} - \ell_{\alpha})/2 = a - b;
\]
so the matrix exponents are zero, and we get
\[
\delta_{\text{new}} = [I, I, e(-t_{Z}/2)] \delta_0
\]

(2.8m) \{e:deltarelation\}

We can now complete the proof of Lemma 2.1.

According to (2.3j), the coefficient we want is $-1$ if $J(E)$ is a composition factor of $I(F_{\text{new}})$, and $+1$ otherwise. According to Lemma 2.4(4) this occurrence as a composition factor depends on the agreement of two characters of $(h_{\text{new}}H)^{\xi} \cap (h_{\text{new}}H)^{\xi}$. The two maximal tori $H$ and $H_{\text{new}}$ together generate $G$, so their intersection must be the center $Z(G)$. So
\[
(H_{\text{new}} \cap H)^{\xi} = Z(G)^{\xi}.
\]

The two characters certainly agree here (for example because the underlying discrete series for $G(\mathbb{R})$ is a composition factor of the principal series for $G(\mathbb{R})$). \{se:endproof\}

The other coset is represented by the element $\delta_{\text{new}}$; so the question we must finally answer is

\[
\text{do the characters } \Lambda(E) \otimes \omega(\alpha, \beta) \text{ and } \Lambda(F_{\text{new}}) \text{ agree on } \delta_{\text{new}}? \quad (2.9a)
\]

Part of the definition of $\Lambda(F_{\text{new}})$ is that
\[
\Lambda(F_{\text{new}})(\delta_{\text{new}}) = z(F), \quad (2.9b)
\]

and similarly
\[
\Lambda(E)(\delta) = z(E). \quad (2.9c)
\]

The factor in square brackets in (2.8m) belongs to the identity component of the -1 eigenspace of $\delta$ on $H^{\xi}$, so the $\delta$-fixed characters $\lambda$ and $\omega(\alpha, \beta)$ must be trivial on it:
\[
\Lambda(E) \otimes \omega(\alpha, \beta) \left( \begin{bmatrix} i & 0 & 0 \\ 0 & i^{-1} & 0 \end{bmatrix} t_{\alpha} , \begin{bmatrix} i & 0 & 0 \\ 0 & i^{-1} & 0 \end{bmatrix}^{-t_{\alpha}}, 1 \right) = 1. \quad (2.9d)
\]

Applying (2.6), we get
\[
\Lambda(E) \otimes \omega(\alpha, \beta)(\delta_{\text{new}}) = -z(E). \quad (2.9e)
\]

We get occurrence as a composition factor, and so a coefficient of $-1$ in $T_{\kappa}$, if and only if $z(F)/z(E) = -1$. \qed
The goal here is to look at the 1i cases to see whether there are problems with the formulas from [2].

**Lemma 3.1.** Suppose $\alpha$ is of type 1i* for $E = (\lambda, \tau, \ell, t)$. Define

$$\ell_{\text{split}} = \ell + \left(\frac{g_\alpha - \ell_\alpha - 1}{2}\right)\alpha^\vee.$$ 

Suppose that

$$F = (\lambda', \tau', \ell_{\text{split}}, t)$$

is an extended parameter of type 1r* appearing in $T_\alpha(E)$. Then the coefficient with which it appears is the ratio of the $z$-values for these two extended parameters (see [2, Definition 5.5]). Explicitly, this is

$$z(\lambda', \tau', \ell_{\text{split}}, t)/z(\lambda, \tau, \ell, t) = i^{\left(\tau', (\delta-1)\ell_{\text{split}}\right) - \left(\tau, (\delta-1)\ell\right)} (-1)^{(\lambda'-\lambda, t)}.$$ 

**Proof.** As mentioned in the introduction, the definition of $T_\alpha$ involves sheaves on a form of $G$ defined over a finite field. It is very easy to see from that definition that one can make the computation entirely in the Levi subgroup of $G$ defined by

$$\alpha = \vee \delta(\alpha).$$

(3.2a)

We may therefore assume that $G$ is equal to $L$. Writing $Z$ for the identity component of the center of $G$, this means that

$G$ is a quotient of $SL(2) \times Z$ 

(3.2b)

by a finite central subgroup. Accordingly there is a natural identification of Lie algebras

$$\mathfrak{g} = \mathfrak{sl}(2) \times \mathfrak{z}.$$ 

(3.2c)

We use the standard torus

$$H = \left\{ \left[ \begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right], z \mid x, z \in \mathbb{C}^\times, z \in \mathbb{Z} \right\}$$

(3.2d)

(Note that $H$ is a quotient of $\mathbb{C}^\times \times \mathbb{Z}$, not a direct product.) The Lie algebra of $H$ is identified in this way as

$$\mathfrak{h} \simeq \mathbb{C} \times \mathfrak{z}, \quad L \mapsto (\alpha(L)/2, L_Z) = (L_\alpha/2, L_Z);$$

(3.2e)

here $L_Z$ is the projection of $L$ on $\mathfrak{z}$. The simple coroot is

$$H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot 0 = (1, 0, 0).$$

(3.2f)
The pinning is given by the simple root vector
\[ X_\alpha = \left[ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right], 0 \] (3.2g)

The Tits group generator is
\[ \sigma_\alpha = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right], 1 \] (3.2h) {e:oneSL2tits}

Here is the strategy of the proof. The terms \( \ell \) and \( \ell^{\text{split}} \) in our extended parameters define strong involutions \( \xi \) and \( \xi^{\text{split}} \), and therefore subgroups
\[ K_\xi = G^\xi, \quad K^{\text{split}} = G^{\xi^{\text{split}}}. \] (3.3a)

These have index two in the corresponding subgroups of the extended group
\[ \delta_0 K_\xi = [\delta_0 G]^\xi, \quad \delta_0 K^{\text{split}} = [\delta_0 G]^{\xi^{\text{split}}}. \] (3.3b) {e:extK1}

The hypothesis that \( F \) appears in \( T_\alpha(E) \) means in particular that \( \xi^{\text{split}} \) is conjugate to \( \xi \) by a unique coset \( gK_\xi \).

The extended parameters \( E \) and \( F \) define
\[ J(E) = \text{irreducible } (\mathfrak{g}, \delta_0 K_\xi)-\text{module} \]
\[ I(F) = \text{standard } (\mathfrak{g}, \delta_0 K^{\text{split}})-\text{module} \]
\[ I(F)^{\text{new}} = \text{standard } (\mathfrak{g}, \delta_0 K_\xi)-\text{module}; \] (3.3c)

the last is obtained by twisting \( I(F) \) by \( \text{Ad}(g) \). The representation-theoretic interpretation of the results of [5] says that
\[ \text{coeff. of } F \text{ in } T_\alpha(E) = m(J(E), I(F)^{\text{new}}) - m(J(E), I(-F)^{\text{new}}). \] (3.3d) {e:Talphamult}

Here \( I(-F^{\text{new}}) \) means \( I(F^{\text{new}}) \) tensored with the nontrivial character of \( \delta_0 G/G \), the other extension of the standard representation to the extended group; and \( m(\cdot, \cdot) \) denotes multiplicity as a composition factor. It turns out that exactly one of these multiplicities is nonzero, and that one is 1; so determining the sign of \( F \) in \( T_\alpha(E) \) means determining whether or not \( J(E) \) appears in \( I(F)^{\text{new}} \).

Up to this point, the reduction to \( SL(2) \) is unimportant: we could have said exactly the same words on the original larger group \( G \). But our determination of the multiplicity will use special facts about \( SL(2) \). Here they are.

**Lemma 3.4.** Suppose we are in the setting (3.2).

1. The discrete series \( (\mathfrak{g}, \delta_0 K_\xi)-\text{module} \) \( J(E) \) is uniquely determined by its infinitesimal character and (unique) lowest \( \delta_0 K_\xi \)-type.

2. If we define
\[ \delta_0 K_\xi^\# = \langle K_\xi^0, (\delta_0 H)\xi \rangle = (\delta_0 H)\xi, \]
then this lowest $δ_0 K_ξ$-type is

$$\text{Ind}_{δ_0 K_ξ}^{δ_0 K_ξ^0} (Λ(E) ⊗ α)$$

Here $Λ(E)$ is the character of the extended torus $(δ_0 H)^ξ$ defined by $E$, and $α$ means the character by which $δ_0 H$ acts on $X_α$.

3. Write $H^{new} = \text{Ad}(g)(H)$, with $g$ defined after (2.3b), and $Λ(F^{new})$ for the corresponding one-dimensional character of $(δ_0 H^{new})^ξ$. Then

$$I(F^{new})|_{δ_0 K_ξ} = \text{Ind}_{(δ_0 H^{new})^ξ}^{δ_0 K_ξ^0} (Λ(F^{new})).$$

4. The discrete series representation $J(E)$ is a composition factor of the principal series representation $I(F^{new})$ if and only if

$$\text{Hom}_{(δ_0 H^{new})^ξ \cap (δ_0 H)^ξ} (Λ(E) ⊗ α, Λ(F^{new})) \neq 0.$$

Proof. Part (1) is a well-known general fact about discrete series representations for reductive groups; the extension to $δ_0$-fixed discrete series for extended groups is routine. Part (2) is equally general; for general $G$ or $δ_0 G$ the inducing representation is the lowest $K_ξ^#$ or $δ_0 K_ξ^#$-type. Part (3) is a general fact about principal series representations attached to split maximal tori; we have just inserted the value of $2ρ$ for our $G$.

For (4), because the infinitesimal characters of $J(E)$ and $I(F^{new})$ are both given by the (unwritten) parameter $γ$, we just need (by (1)) to determine whether the lowest $δ_0 K_ξ$-type of $J(E)$ appears in $I(F^{new})$. Using (2), this amounts deciding the nonvanishing of

$$\text{Hom}_{(δ_0 H^{new})^ξ \cap (δ_0 H)^ξ} (Λ(E) ⊗ α, Λ(F^{new})) \neq 0.$$ (3.5a)

Because $δ_0 K_ξ^#$ meets both cosets of the inducing subgroup in (3), we get

$$I(F^{new})|_{δ_0 K_ξ^#} = \text{Ind}_{(δ_0 H^{new})^ξ \cap δ_0 K_ξ^#}^{δ_0 K_ξ^#} (Λ(F^{new})).$$ (3.5b)

Another application of Frobenius reciprocity says that we are left with deciding the nonvanishing of

$$\text{Hom}_{(δ_0 H^{new})^ξ \cap (δ_0 H)^ξ} (Λ(E) ⊗ α, Λ(F^{new})),$$ (3.5c)

as we wished to show.

In order to prove Lemma 3.1, we will write down everything explicitly, in order to compute $(δ_0 H^{new})^ξ \cap (δ_0 H)^ξ$ and determine whether the two characters agree there.
Write $\xi_{0,Z}$ and $\delta_{0,Z}$ for the restrictions to $Z$ of the (commuting) distinguished involutions of $[2, (11a)]$; then

$$\xi_{0}(g,z) = (g,\xi_{0,Z}(z)), \quad \delta_{0}(g,z) = (g,\delta_{0,Z}(z)). \quad (3.6a)$$

(Here (and below) we have imprecisely written $(g,z)$ to mean on the left (of each formula in (3.6a)) a choice of preimage in $SL(2) \times Z$ of an element of $G$, and on the right the image in $G$. Another way to make the formulas precise is to note that the automorphisms $\xi_{0}$ and $\delta_{0}$ lift uniquely to $SL(2) \times Z$.)

We are concerned with multiplying $\xi_{0}$ and $\delta_{0}$ by torus elements (and, eventually, Tits group elements). This involves the map

$$e(L) = \exp(2\pi iL) : g \to G. \quad (3.6b)$$

For $L \in h$, in the coordinates of (3.2e), this is

$$e(L) = (\exp(\pi iL_{\alpha}), e(L_{Z})). \quad (3.6c)$$

If $L$ is half-integral (so that $2L_{\alpha}$ is an integer) this is

$$e(L) = \left[ \begin{array}{cc} i^{2L_{\alpha}} & 0 \\ 0 & i^{-2L_{\alpha}} \end{array} \right] e(L_{Z}). \quad (3.6d)$$

The strong involution of $G$ attached to our extended parameter $E$ is

$$\xi = e((g - \ell)/2)\xi_{0}
= \left[ \begin{array}{cc} i^{g_{\alpha} - \ell_{\alpha}} & 0 \\ 0 & i^{-(g_{\alpha} - \ell_{\alpha})} \end{array} \right] e((gZ - \ellZ)/2) \xi_{0}. \quad (3.6e)$$

Because $g_{\alpha} - \ell_{\alpha}$ is odd (this is the “$i$” part of the nature of our extended parameter) the conclusion is that

$$\xi \text{ acts on the } SL(2) \text{ factor by conjugation by } \left[ \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right]. \quad (3.6f)$$

In particular, the action on the standard torus $\mathbb{C}^{\times}$ is trivial; so

$$H^{\xi} = \mathbb{C}^{\times} \times Z^{\xi_{0}}. \quad (3.6g)$$

The extended parameter $E$ provides also a representative

$$\delta = e(-t/2)\delta^{0} = \left[ \begin{array}{cc} i^{-t_{\alpha}} & 0 \\ 0 & i^{t_{\alpha}} \end{array} \right] e(-tZ/2) \delta_{0} \quad (3.6h)$$

for the other coset of $(\delta_{0}H)^{\xi}$. The defining equation $(1 + \theta)t = (\delta - 1)\ell$ tells us that $t_{\alpha} = 0$, so

$$\delta = e(-t/2)\delta^{0} = [I, e(-tZ/2)] \delta_{0}. \quad (3.6i) \quad (e:deltaonecpt)$$

Now it is clear (because we are just going to be conjugating by $SL(2)$) that this element $\delta$ is also the representative defined by $F$ for $H^{split}$ and for $H^{new}$.
\[ \delta_{\text{new}} = [I, e(-tZ/2)] \delta_0 = \delta. \]  

(3.6j) \{e:deltaonerelation\}

We can now complete the proof of Lemma 3.1. According to (3.3d), the coefficient we want is +1 if \( J(E) \) is a composition factor of \( I(F_{\text{new}}) \), and \(-1\) otherwise. According to Lemma 3.4(4) this occurrence as a composition factor depends on the agreement of two characters of \( (\delta_0 H_{\text{new}})^\xi \cap (\delta_0 H)^\xi \). The two maximal tori \( H \) and \( H_{\text{new}} \) together generate \( G \), so their intersection must be the center \( Z(G) \). So
\[ (H_{\text{new}} \cap H)^\xi = Z(G)^\xi. \]

The two characters certainly agree here (for example because the underlying discrete series for \( G(\mathbb{R}) \) is a composition factor of the principal series for \( G(\mathbb{R}) \)).

The other coset is represented by the element \( \delta_{\text{new}} \); so the question we must finally answer is whether or not the two characters \( \Lambda(E) + \alpha \) and \( \Lambda(F_{\text{new}}) \) agree on \( \delta_{\text{new}} = \delta \). Because the character \( \alpha \) is trivial on \( \delta \), the character \( \Lambda(E) + \alpha \) takes the value \( z(E) \) on \( \delta_{\text{new}} \). In the same way the character \( \Lambda(F_{\text{new}}) \) takes the value \( z(F) \) on \( \delta_{\text{new}} \). We get occurrence as a composition factor, and so a coefficient of 1 in \( T_\alpha \), if and only if \( z(F)/z(E) = 1 \).

\[ \square \]

References


