Deforming subgroups

This is a sketch of a solution of the homework assigned for March 17, to deform a subgroup $H$ to another $H_0$ of the same dimension, such that $H_0$ is normal in a parabolic subgroup $P$.

Write $I_H \subset k[G]$ for the ideal of the subgroup $H$. I’ll write $\langle E \rangle$ for the ideal generated by a set $E$ of functions. Arguing as in the construction of the quotient variety $G/H$ (text, 5.5) we can find a finite-dimensional subspace $V \subset k[G]$ with the following properties.

1) The space $V$ is stable under left translations $\lambda$ and right translations $\rho$:
   
   \begin{align*}
   [\lambda(g)f](x) &= f(g^{-1}x), \\
   [\rho(g)f](x) &= f(xg).
   \end{align*}

2) The ideal $I_H$ is generated by its intersection with $V$:
   
   $I_H(1) = I_H \cap V$, \quad $I_H = \langle I_H(1) \rangle$.

Here is the approximate idea. The ideal of the subgroup $gHg^{-1} = \text{Ad}(g)H$ is

$I_{gHg^{-1}} = \text{Ad}(g)I_H = \langle \text{Ad}(g)I_H(1) \rangle$.

The action of $\text{Ad}(g)$ on functions is $\lambda(g)\rho(g)$, so it preserves the finite-dimensional space $V$. We consider the action of $\text{Ad}$ on the (projective) Grassmann variety $\text{Gr}_d(V)$ of $d$-dimensional subspaces of $V$ (with $d = \dim I_H(1)$). The ideals of the conjugates of $H$ form a single $G$-orbit. The isotropy group for this action is clearly the normalizer of $H$:

$\{gHg^{-1} \mid g \in G\} \simeq \text{Ad}(G)I_H(1) \simeq G/N_G(H) \subset \text{Gr}_d(V)$.

In the closure of this $G$-orbit there must be a closed $G$ orbit $\text{Ad}(g)W(1)$, with $W(1)$ a $d$-dimensional subspace of $V$. Since the Grassmannian is projective, this closed orbit is complete, so its isotropy group $P(1)$ is parabolic.

Approximately $H_0$ should be the subgroup defined by the ideal $\langle W(1) \rangle$ generated by $W(1)$.

The difficulty is that this ideal may be much smaller than you expect (so that the dimension of the corresponding variety is larger). Even though $W(1)$ is a limit of the corresponding part of the ideals for conjugates of $H$, it may not be true that the entire ideals (for conjugates of $H$) converge to $\langle W(1) \rangle$.

I don’t know a really simple way to fix matters, but here is something that seems to work. We may (after enlarging $V$) assume that it generates $k[G]$. We may then filter $k[G]$ by defining

$k[G]_m = \text{span of products of at most } m \text{ factors in } V$,

so that

$k[G]_0 = k \subset k[G]_1 = k + V \subset k[G]_2 \subset \ldots$

This is an exhaustive increasing filtration which respects multiplication:

$k[G]_m \cdot k[G]_n \subset k[G]_{m+n}$,
and the associated graded ring is a finitely generated commutative algebra over $k$ (the quotient of a polynomial ring in $\dim V$ variables by a homogeneous ideal).

Now we can define

$$I_H(m) = I_H \cap k[G]_m, \quad \dim I_H(m) = d_m.$$ 

Define a “partial flag variety”

$$X(m) = \{ \text{chains of subspaces } W = \{ W(1) \subset W(2) \subset \cdots \subset W(m) \subset k_m[G] \} \}$$

subject to the requirements

$$\dim W(j) = d_j, \quad W(j) \subset k[G].$$

This is a projective algebraic variety. The choice of $V$ and the construction of the filtration makes $k[G]_m$ stable by $\rho, \lambda$, and $\text{Ad}$, so these actions apply to $X(m)$. Forgetting the largest subspace defines a proper morphism

$$\pi(m + 1): X(m + 1) \to X(m).$$

Inside $X(m)$ is the $G$-orbit

$$Z(m) = \text{Ad}(G)I_H.$$ 

Since by construction $I_H(1)$ generates $I_H$, it is very easy to check that all the isotropy groups

$$\{ g \in G \mid \text{Ad}(g)I_H(i) = I_H(i), 1 \leq i \leq m \}$$

are equal to $N_G(H)$; so

$$Z(m) \simeq G/N_G(H), \quad m \geq 1.$$ 

Each closure $\overline{Z(m)}$ is $\text{Ad}(G)$-stable and closed in the projective variety $X(m)$, and therefore complete. I am going to choose a compatible family of closed $G$ orbits

$$O(m) = \text{Ad}(G)W_0 \subset \overline{Z(m)}.$$ 

The notation is a little ambiguous. The flag $W_0$ in the preceding formula consists of $m$ subspaces

$$W_0(i) \quad (1 \leq i \leq m);$$

but the $m$ does not appear in the notation. When I choose another flag $W'_0$ in $\overline{Z(m + 1)}$, its first $m$ subspaces $W'_0(i)$ will be equal to $W_0(i)$. So calling the new flag $W_0$ is more or less harmless.

We have already seen that $\pi(m + 1)$ maps $Z(m + 1)$ isomorphically onto $Z(m)$; so

$$\pi(m + 1): \overline{Z(m + 1)} \to \overline{Z(m)}$$

is a surjective proper map.
Begin by choosing a closed orbit
\[ O(1) = \text{Ad}(G)W_0(1) \subset Z(1). \]

Once the closed orbit
\[ O(m) = \text{Ad}(G)W_0 \subset Z(m) \]
is chosen, its preimage
\[ \pi(m+1)^{-1}(O(m)) \subset Z(m+1) \]
is necessarily a complete subvariety, preserved by Ad (since \( \pi(m+1) \) is proper and respects all the group actions). Consequently there is a closed orbit
\[ O(m+1) = \text{Ad}(G)W_0' \subset \pi(m+1)^{-1}(O(m)). \]

We may choose the orbit representative \( W_0' \) to project to \( W_0 \), which means exactly that the first \( m \) subspaces in the flag \( W_0' \) agree with those already chosen.

Because the orbits are closed, the isotropy groups
\[ P(m) = \{ g \in G \mid \text{Ad}(g)W_0(i) = W_0(i)(1 \leq i \leq m) \} \]
are all parabolic.

We now consider the increasing family of ideals
\[ I_0(m) = \langle W_0(m) \rangle \subset k[G]. \]
Because \( k[G] \) is Noetherian, this family is eventually constant:
\[ I_0(m) = I_0(M) \quad (m \geq M). \]

We call this limiting ideal \( I_0 \), and define
\[ H_0 = \text{variety of } I_0. \]

Because \( I_0 \) was constructed as a limit of ideals, it is easy to check that
\[ I_0 \cap k[G]_m = W_0(m) \quad (1 \leq m < \infty), \]
and in particular that
\[ \dim(I_0 \cap k[G]_m) = \dim(I_H \cap k[G]_m) \quad (1 \leq m < \infty). \]

From this knowledge of Hilbert functions we conclude that \( H_0 \) has the same dimension as \( H \).

We want to show that \( H_0 \) is a group. Recall the product morphism
\[ \mu: G \times G \to G, \quad \mu(x,y) = xy, \]
and the corresponding algebra homomorphism

\[ \mu^*: k[G] \rightarrow k[G] \otimes k[G]. \]

If \( A, B, \) and \( C \) are closed subsets of \( G \), with ideals \( I_A, I_B, \) and \( I_C, \) then the ideal of \( A \times B \)

is \( I_A \otimes k[G] + k[G] \otimes I_B; \) so the condition \( A \cdot B \subseteq C \) is equivalent to

\[ \mu^*(I_C) \subseteq I_A \otimes k[G] + k[G] \otimes I_B. \]

In particular, the condition that \( H \) is closed under multiplication is

\[ \mu^*(I_H) \subseteq I_H \otimes k[G] + k[G] \otimes I_H. \]

For every positive integer \( m, \) the subspace \( I_H(m) \) is finite-dimensional; so there must be

positive \( N(m) \) so that

\[ \mu^*(I_H(m)) \subseteq I_H(N) \otimes k[G]^N + k[G]^N \otimes I_H(N). \]

Because this condition (concerning behavior of subspaces under the fixed linear map \( \mu^* \)

on fixed finite-dimensional spaces like \( k[G]^m \)) is satisfied for all the ideals (and corresponding

flags) \( I_{gHg^{-1}} \), it is satisfied by the limit ideal \( I_0 \) as well. It follows that \( H_0 \) is closed under

multiplication.

Similar arguments show that \( H_0 \) is closed under inversion and contains the identity

of \( G, \) so \( H_0 \) is a subgroup.

The ideal \( I_0 \) may not be radical, so the ideal of \( H_0 \) may be slightly larger than \( I_0; \) but

at any rate it is clear that \( N_G(H_0) \) contains the isotropy group \( P(M) \) of the closed orbit

\( O(M). \) Since \( P(M) \) is parabolic, the larger group \( N_G(H_0) \) is parabolic as well.