Inflatable mathematics

David Vogan

Sophus Lie Days, Cornell, April 27, 2008
Outline

Building up from simple pieces

Ideas from linear algebra

Bruhat order

Schubert varieties

Calculating with(out) Schubert varieties

Kazhdan-Lusztig polynomials

An addiction to silicon
The main idea

Begin with **linear algebra**: solving systems of linear equations by Gaussian elimination.
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Idea: *reduce number of coordinates by one.*
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Idea: **reduce number of coordinates by one**.

Relate to **geometry**: arranging lines and planes.
The main idea

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Relate to geometry: arranging lines and planes.

Idea: reduce to geometry of one dimension less.
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The main idea

Begin with **linear algebra**: solving systems of linear equations by Gaussian elimination.
Idea: *reduce number of coordinates by one*.
Relate to **geometry**: arranging lines and planes.
Idea: *reduce to geometry of one dimension less*.

Use same idea for more complicated geometry.
Gaussian elimination: easy cases

System of three equations in three unknowns is

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3
\end{align*}
\]

I’ll assume always the system has just one solution.
Gaussian elimination: easy cases

System of three equations in three unknowns is

\[ a_{11} x_1 = c_1 \]
\[ a_{22} x_2 = c_2 \]
\[ a_{33} x_3 = c_3 \]

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Easiest case is diagonal system: divide each equation by a constant to solve.
Gaussian elimination: easy cases

System of three equations in three unknowns is

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\begin{align*}
    a_{11} x_1 & = c_1 \\
    a_{21} x_1 + a_{22} x_2 & = c_2 \\
    a_{31} x_1 + a_{32} x_2 + a_{33} x_3 & = c_3
\end{align*}
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Next easiest is lower triangular: add multiples of some eqns to later ones to make diagonal.
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Suppose lower triangular EXCEPT one coefficient \(a_{12} \neq 0\). Add multiple of 1st eqn to second to get...
Gaussian elimination: easy cases

System of three equations in three unknowns is

\[
\begin{align*}
  a_{11} x_1 + a_{12} x_2 &= c_1 \\
  a_{21}' x_1 &= c_2' \\
  a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= c_3
\end{align*}
\]

I’ll assume always the system has just one solution.

Easiest case is diagonal system: divide each equation by a constant to solve.

Next easiest is lower triangular: add multiples of some eqns to later ones to make diagonal.

Suppose lower triangular EXCEPT one coefficient \( a_{12} \neq 0 \). Add multiple of 1st eqn to second to get…

This system is nearly lower triangular, except that the first two equations are interchanged.
“Typical” system of equations in three unknowns is

\[
\begin{align*}
    a_{11} x_1 & + a_{12} x_2 + a_{13} x_3 = c_1 \\
    a_{21} x_1 & + a_{22} x_2 + a_{23} x_3 = c_2 \\
    a_{31} x_1 & + a_{32} x_2 + a_{33} x_3 = c_3
\end{align*}
\]

where “typically” \( a_{13} \neq 0 \). Add multiple of 1st equation to each later eqn to get…
Gaussian elimination: typical case

“Typical” system of equations in three unknowns is

\[
\begin{align*}
 a_{11} x_1 &+ a_{12} x_2 + a_{13} x_3 = c_1 \\
 a_{21}' x_1 &+ a_{22}' x_2 = c_2' \\
 a_{31}' x_1 &+ a_{32}' x_2 = c_3'
\end{align*}
\]

where “typically” \( a_{13} \neq 0 \). Add multiple of 1st equation to each later eqn to get.

Now “typically” \( a_{22}' \neq 0 \). Add multiple of 2nd eqn to last to get. . .
Gaussian elimination: typical case

“Typical” system of equations in three unknowns is

\[
\begin{align*}
    a_{11} x_1 & + a_{12} x_2 + a_{13} x_3 = c_1 \\
    a_{21} x_1' & + a_{22} x_2' = c_2' \\
    a_{31}'' x_1 & = c_3''
\end{align*}
\]

where “typically” \(a_{13} \neq 0\). Add multiple of 1st equation to each later eqn to get. . .

Now “typically” \(a_{22}' \neq 0\). Add multiple of 2nd eqn to last to get. . .

Again this last system is nearly lower triangular, except that order of the three eqns is reversed.
Gaussian elimination: typical case

“Typical” system of equations in three unknowns is

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\
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Again this last system is nearly lower triangular, except that order of the three eqns is reversed.

To say what happens in general, use matrix notation \( Ax = c \). Here \( A = (a_{ij}) \) is \( n \times n \) coeff matrix, and \( x = (x_j) \) is the column vector of \( n \) unknowns.
Theorem for Gaussian elimination

Theorem
Theorem for Gaussian elimination

Theorem

Suppose $A$ is an invertible $n \times n$ matrix, and $c$ is an $n$-tuple of constants. Consider the system of $n$ equations in $n$ unknowns

$$Ax = c.$$
Theorem for Gaussian elimination

**Theorem**

Suppose $A$ is an invertible $n \times n$ matrix, and $c$ is an $n$-tuple of constants. Consider the system of $n$ equations in $n$ unknowns

$$Ax = c.$$

*Using the two operations*

1. dividing an equation by a non-zero constant, and
2. adding a multiple of one equation to a later one,
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Using the two operations

1. dividing an equation by a non-zero constant, and
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we can transform this system into a new one

$$A'x = c'.$$
Theorem for Gaussian elimination

Theorem

Suppose $A$ is an invertible $n \times n$ matrix, and $c$ is an $n$-tuple of constants. Consider the system of $n$ equations in $n$ unknowns

$$Ax = c.$$ 

Using the two operations

1. dividing an equation by a non-zero constant, and
2. adding a multiple of one equation to a later one,

we can transform this system into a new one

$$A'x = c'.$$

The new system, after reordering the equations, is lower triangular.
Possibilities for three unknowns

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
123
\end{pmatrix}
\]

\[a_{12} = a_{23} = a_{13} = 0\]
Possibilities for three unknowns

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
(123)
\]

\[
a_{12} = a_{23} = a_{13} = 0
\]
Possibilities for three unknowns

\begin{align*}
\left( \begin{array}{ccc} *
& 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \end{array} \right) \quad & \text{(213)} & \left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & * & 1 \\
0 & 0 & 1 \end{array} \right) \quad & \text{(132)} \\
a_{13} = a_{23} = 0, \ a_{12} \neq 0 & & a_{12} = a_{13} = 0, \ a_{23} \neq 0 \\
\left( \begin{array}{ccc} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{array} \right) \quad & \text{(123)} & & \\
a_{12} = a_{23} = a_{13} = 0
\end{align*}
Possibilities for three unknowns

\[
\begin{pmatrix}
* & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{(213)}
\]
\[a_{13} = a_{23} = 0, \quad a_{12} \neq 0\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & * & 1 \\
0 & 1 & 0
\end{pmatrix} \quad \text{(132)}
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\[a_{12} = a_{13} = 0, \quad a_{23} \neq 0\]

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\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
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\end{pmatrix} \quad \text{(123)}
\]
\[a_{12} = a_{23} = a_{13} = 0\]
Possibilities for three unknowns

\[
\begin{pmatrix}
* & 1 & 0 \\
* & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} \quad (312)
\]
a_{13} = 0, \ a_{12} \neq 0, \ a_{23} \neq 0

\[
\begin{pmatrix}
* & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad (213)
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a_{13} = a_{23} = 0, \ a_{12} \neq 0

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\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
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\end{pmatrix} \quad (123)
\]
a_{12} = a_{23} = a_{13} = 0

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0 & 1 & 0 \\
0 & 0 & 1
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\]
a_{12} = a_{13} = 0, \ a_{23} \neq 0

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1 & 0 & 0 \\
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\quad (213)
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\[a_{13} = a_{23} = 0, \ a_{12} \neq 0\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
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\end{pmatrix}
\quad (123)
\]

\[a_{12} = a_{23} = a_{13} = 0\]
Possibilities for three unknowns

\[
\begin{pmatrix}
* & 1 & 0 \\
* & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]
\begin{align*}
\text{(312)} & \quad a_{13} = 0, \quad a_{12} \neq 0, \quad a_{23} \neq 0 \\
\text{(321)} & \quad a_{13} \neq 0, \quad \left| \begin{array}{cc}
a_{12} & a_{13} \\
a_{22} & a_{23}
\end{array} \right| \neq 0 \\
\end{align*}

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\begin{align*}
\text{(123)} & \quad a_{12} = a_{23} = a_{13} = 0 \\
\text{(132)} & \quad a_{12} = a_{13} = 0, \quad a_{23} \neq 0 \\
\text{(231)} & \quad a_{12} \neq 0, \quad a_{13} \neq 0 \\
\text{(213)} & \quad a_{13} = 0, \quad a_{12} \neq 0, \quad a_{23} \neq 0 \\
\end{align*}
From algebra to geometry

A flag in 3 dimensions is a (straight) line through the origin, contained inside a plane through the origin:
From algebra to geometry

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One flag not so interesting. What’s interesting is how many different flags there are, and how they’re related.
From algebra to geometry

A flag in 3 dimensions is a (straight) line through the origin, contained inside a plane through the origin:

One flag not so interesting. What’s interesting is how many different flags there are, and how they’re related.

System of equations $= 3 \times 3$ matrix $\rightsquigarrow$ flag:
line $= \text{multiples of first row}$, plane $= \text{span of first two rows}$. 
From algebra to geometry

A **flag** in 3 dimensions is a (straight) line through the origin, contained inside a plane through the origin:

![Flag example](image)

One flag not so interesting. What’s interesting is how many different flags there are, and how they’re related.

**System of equations** $= 3 \times 3$ matrix $\mapsto$ **flag**:
- line $=$ multiples of first row,
- plane $=$ span of first two rows.

*Two matrices give same flag if and only if differ by*
- *multiply row by constant*
- *add multiple of one row to later row.*
Possible flags $L \subset P$

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad
(123)
\]

$L_x \subset P_{xy}$
Possible flags $L \subset P$

$\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \quad (123)$

$L_x \subset P_{xy}$
Possible flags $L \subset P$

\[
\begin{pmatrix}
* & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad (213) \quad L_x \not= L \subset P_{xy}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & * & 1 \\
0 & 1 & 0
\end{pmatrix} \quad (132) \quad L_x \subset P \not= P_{xy}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad (123) \quad L_x \subset P_{xy}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad (321) \quad P_{xy} \not\supset L \subset P
\]

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\begin{pmatrix}
1 & 0 & 0 \\
0 & * & 1 \\
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\]

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Possible flags \( L \subset P \)

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\begin{pmatrix}
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\( L_x \not\subset L \subset P_{xy} \)

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\( L_x \subset P \not\subset P_{xy} \)

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\begin{pmatrix}
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\( L_x \subset P \subset P_{xy} \)
Possible flags $L \subset P$

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\begin{pmatrix}
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1 & 0 & 0
\end{pmatrix} \quad (312)
\]

$P_{xy} \supset L \subset P' \not\supset L_x$

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0 & 0 & 1
\end{pmatrix} \quad (213)
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$L_x \not\supset L \subset P_{xy}$

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\begin{pmatrix}
* & * & 1 \\
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0 & 1 & 0
\end{pmatrix} \quad (231)
\]

$P_{xy} \supset L' \subset P \supset L_x$

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & * & 1 \\
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\end{pmatrix} \quad (132)
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$L_x \subset P \not= P_{xy}$

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0 & 1 & 0 \\
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Possible flags $L \subset P$

$P_{xy} \supset L \subset P' \not\supset L_x$

$L_x \not\subset L \subset P_{xy}$

$L_x \subset P \not\supset P_{xy}$

$P_{xy} \supset L' \subset P \supset L_x$

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P_{xy} \not\supset L \subset P' \not\supset L_x
\]

\[
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\]

\[
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P_{xy} \not\supset L \subset P \not\supset P_{xy}
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P_{xy} \not\supset L \subset P \not\supset P_{xy}
\]

\[
P_{xy} \not\supset L \subset P \not\supset P_{xy}
\]
Geometric picture

Moving up $\leadsto$ more complicated geometry.
Geometric picture

Moving up $\rightsquigarrow$ more complicated geometry.
up one blue step: fixed line $\rightsquigarrow$ variable line in a plane.
Geometric picture

Moving up $\sim$ more complicated geometry.
up one blue step: fixed line $\sim$ variable line in a plane.
up one red step: fixed plane $\supset L \sim$ variable plane $\supset L$. 
Geometric picture

Moving up $\leadsto$ more complicated geometry.
up one **blue step**: fixed line $\leadsto$ variable line in a plane.
up one **red step**: fixed plane $\supset L \leadsto$ variable plane $\supset L$.
**Inflatable mathematics**: replace points by circles.
Geometric picture

Moving up $\leadsto$ more complicated geometry.

up one **blue step**: fixed line $\leadsto$ variable line in a plane.

up one **red step**: fixed plane $\supseteq L$ $\leadsto$ variable plane $\supseteq L$.

**Inflatable mathematics**: replace points by circles.
Geometric picture

Moving up $\leadsto$ more complicated geometry.
up one blue step: fixed line $\leadsto$ variable line in a plane.
up one red step: fixed plane $\supset L$ $\leadsto$ variable plane $\supset L$.

Inflatable mathematics: replace points by circles.

![Inflatable figure]
Geometric picture

Moving up $\rightsquigarrow$ more complicated geometry.
up one **blue step**: fixed line $\rightsquigarrow$ variable line in a plane.
up one **red step**: fixed plane $\supset L$ $\rightsquigarrow$ variable plane $\supset L$.
**Inflatable mathematics**: replace points by circles.
Geometric picture

Moving up $\sim$ more complicated geometry.
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Inflatable mathematics: replace points by circles.
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Moving up $\rightsquigarrow$ more complicated geometry.
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Inflatable mathematics: replace points by circles.
Geometric picture

Moving up \( \rightsquigarrow \) more complicated geometry.
up one **blue** step: fixed line \( \rightsquigarrow \) variable line in a plane.
up one **red** step: fixed plane \( \supset L \rightsquigarrow \) variable plane \( \supset L \).
**Inflatable mathematics:** replace points by circles.

\[
L_x \subset P_{xy}
\]
**Geometric picture**

Moving up \( \sim \) more complicated geometry.
up one **blue step**: fixed line \( \sim \) variable line in a plane.
up one **red step**: fixed plane \( \supset L \sim \) variable plane \( \supset L \).

**Inflatable mathematics**: replace points by circles.

\[ L_x \subset P_{xy} \]

- **Inflatable mathematics**
- Introduction
- Gaussian elimination
- Bruhat order
- Schubert varieties
- Calculating with(out) Schubert varieties
- Kazhdan-Lusztig polynomials
- An addiction to silicon
Inflatable mathematics

David Vogan

Introduction
Gaussian elimination
Bruhat order
Schubert varieties
Calculating with(out) Schubert varieties
Kazhdan-Lusztig polynomials
An addiction to silicon

Geometric picture

Moving up $\sim$ more complicated geometry.
up one **blue step**: fixed line $\sim$ variable line in a plane.
up one **red step**: fixed plane $\supset L$ $\sim$ variable plane $\supset L$.
Inflatable mathematics: replace points by circles.

$L_x \not\subset P = P_{xy}$

$L_x \subset P_{xy}$
Geometric picture

Moving up $\sim$ more complicated geometry.
up one **blue step**: fixed line $\sim$ variable line in a plane.
up one **red step**: fixed plane $\supset L$ $\sim$ variable plane $\supset L$.
**Inflatable mathematics**: replace points by circles.

$L_x \neq L \subset P = P_{xy}$

$L_x \subset P_{xy}$
Geometric picture

Moving up $\rightsquigarrow$ more complicated geometry.

up one **blue step**: fixed line $\rightsquigarrow$ variable line in a plane.

up one **red step**: fixed plane $⊃ L \rightsquigarrow$ variable plane $⊃ L$.

**Inflatable mathematics**: replace points by circles.

$L_x \not\subset P = P_{xy}$

$L_x \subset P \not\subset P_{xy}$

$L_x \subset P_{xy}$
Geometric picture

Moving up $\rightsquigarrow$ more complicated geometry.
up one blue step: fixed line $\rightsquigarrow$ variable line in a plane.
up one red step: fixed plane $\supset L$ $\rightsquigarrow$ variable plane $\supset L$.

Inflatable mathematics: replace points by circles.

$\text{Inflatable mathematics}$

$L_x \not\subset P \subseteq P_{xy}$

$L_x \subset P \neq P_{xy}$

$L_x \subset P_{xy}$
What’s a Schubert variety?

Divided flags (in three dimensions) into six “Bruhat cells” by relation with standard flag \( L_x \subset P_{xy} \).

**Schubert variety** is one cell and everything below it:

\[
\begin{align*}
L' \subset P', L \subset P' \cap P_{xy} & \quad \Rightarrow \quad P_{xy} \nsubseteq L' \subset P' \nsubseteq L_x \\
L \nsubseteq L' \subset P' & \quad \Rightarrow \quad P_{xy} \nsubseteq L_x \\
P_{xy} \nsubseteq L \subset P_x & \quad \Rightarrow \quad L \nsubseteq P_x \nsubseteq L_x \\
L_x \nsubseteq L \subset P_{xy} & \quad \Rightarrow \quad L \nsubseteq L_x \nsubseteq P_{xy} \\
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\end{align*}
\]

What’s almost true: each Schubert variety “inflated” from a smaller one, replacing each point by a circle. Fails only at the top...
What’s a Schubert variety?

Divided flags (in three dimensions) into six “Bruhat cells” by relation with standard flag $L_x \subset P_{xy}$.

**Schubert variety** is one cell and everything below it:

$L' \subset P'$, $L \subset P' \cap P_{xy}$

$P_{xy} \not\supset L' \subset P' \not\supset L_x$

$L_x \not\subset L \subset P_{xy}$

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Fails only at the top...
Mathematics on a need-to-know basis

To compute with Schubert varieties, need only arrangement of blue and red arrows, describing how small Schubert varieties are inflated:

```
(321)

(312)  (231)
↑      ↑
(213)  (132)

(123)
```

Permutations recorded which rows had pivots in Gaussian elimination. Now they’re just symbols.

Rules for making diagram:
1. One entry for each permutation of \{1, 2, 3\}.
2. Exchange 1...2: blue arrow up.
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![Diagram](image)

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|     |     |     |
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|     |     |     |
|     |     |     |
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As many dimensions as you want

Rules in $n$ dimensions:

1. One entry for each permutation of \{1, 2, \ldots, n\}.
2. Exchange $i \ldots i + 1$: arrow up of color $i$.

Counting problems in this picture $\mapsto$ geometry of Schubert varieties.

There are lots of counting games to play…

height of a permutation = $\#\{\text{pairs } (i, j) \text{ out of order}\}$.

$\#\{\text{permutations at height } d\} = \text{coefficient of } x^d \text{ in polynomial}$

$$(1)(1 + x)(1 + x + x^2) \cdots (1 + x + \cdots x^{n-1}).$$

$\#\{\text{ascending paths bottom to top}\} =$

$\binom{n}{2}/1^{n-1}3^{n-2}5^{n-2} \cdots (2n-5)^2(2n-3)$

Stanley’s formula

(Formula says 16 ascending paths bottom to top in this picture.)
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(FORMULA SAYS 16 ASCENDING PATHS BOTTOM TO TOP IN THIS PICTURE.)
More complicated groups

Picture just described (with $n!$ vertices) is for invertible $n \times n$ matrices. This is the basic example of a real reductive Lie group. Mathematicians and physicists look at lots of other reductive groups.

Each reductive group has a finite diagram describing how its big Schubert varieties are “inflated” from smaller ones. This one is for a 45-dimensional group called $SO(5, 5)$.

For this group there are 251 Schubert varieties, but each arrow still means replace points by circles.
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What do you do with the pretty pictures?

Where we started:

systems of $n$ linear eqns \( \overset{\text{Gauss elim}}{\leftrightarrow} \text{group } GL(n) \overset{\leftrightarrow}{\leftrightarrow} \text{Schubert varieties} \overset{\leftrightarrow}{\leftrightarrow} \text{graph with } n! \text{ vertices, arrows of } n - 1 \text{ colors.} \)

Graph tells what cases can happen during Gaussian elimination; how Gaussian elimination changes with the system of equations; even which cases are most common.

Similarly:

math or physics problem \( \overset{\text{reptn theory}}{\leftrightarrow} \text{reductive group } G \overset{\leftrightarrow}{\leftrightarrow} \text{Schubert varieties for } G \overset{\leftrightarrow}{\leftrightarrow} \text{finite graph for inflating.} \)

1979: David Kazhdan (Harvard) and George Lusztig (MIT) showed how to answer questions about representation theory by calculating in the finite graph.

Defined Kazhdan-Lusztig polynomial $P_{x,y}$ for $x$ and $y$ in the graph. Polynomial in $q$, non-neg integer coeffs.

Polynomial is non-zero only if $y$ is above $x$ in graph. Calculated by a recursion based on knowing all $P_{x',y'}$ for $y'$ smaller than $y$. 


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How the computation works

Now fixing a reductive group $G$ and its graph of Schubert varieties.

- For each pair $(x, y)$ of graph vertices, want to compute KL polynomial $P_{x,y}$.
- Induction: start with $y$’s on bottom of graph, work up. For each $y$, start with $x = y$, work down.
- Seek line $x$ same color as some line $y$.

If it’s there, then $P_{x,y} = P_{x',y}$ (known by induction).
If not, $(x, y)$ is primitive: no color down from $y$ goes up from $x$.

- One hard calculation for each primitive pair $(x, y)$. 
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  $|$ 

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- For each pair $(x, y)$ of graph vertices, want to compute KL polynomial $P_{x,y}$.
- Induction: start with $y$’s on bottom of graph, work up. For each $y$, start with $x = y$, work down.
  \[ x' \]
  \[ y' \]
- Seek line up $x$ same color as some line down $y$.

  If it’s there, then $P_{x,y} = P_{x',y}$ (known by induction).
  If not, $(x, y)$ is primitive: no color down from $y$ goes up from $x$.

- One hard calculation for each primitive pair $(x, y)$. 
What to do for primitive pair \((x, y)\)

- graph vertex \(y \leftrightarrow \text{big Schubert variety } F_y\).
- lower vertex \(x \leftrightarrow \text{little Schubert variety } F_x\).
  
  \(P_{x,y}\) describes how \(F_y\) looks near \(F_x\).

- Pick line down \(y\); means \(F_y \approx \text{inflated from } F_{y'}\).

- Primitive means red line \(x\) is also down from \(x\).

- Geometry translates to algebra \(P_{x,y} \approx P_{x',y'} + qP_{x,y'}\). Precisely:

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P_{x,y} = P_{x',y'} + qP_{x,y'} - \sum_{x' \leq z < y'} \mu(z, y')q^{l(y')-l(z)-1}/2 P_{x',z}.
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Forming the *Atlas* group

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In 2001, Jeff Adams at University of Maryland decided computers and mathematics had advanced far enough to begin interesting work in that direction.

Adams formed a research group *Atlas of Lie groups and representations*, aimed in part at producing software to make old mathematics widely accessible, and to find new mathematics.

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How do you make a computer do that?

▶ In June 2002, Jeff Adams asked Fokko du Cloux.
▶ In November 2005, Fokko finished the program.

Wasn’t that easy?

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Big unknown: number of distinct polynomials. Hoped 400 million polys $\sim 75$G total RAM. Feared 1 billion $\sim 150$G total RAM.
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12/03/06 Marc van Leeuwen made Fokko’s code modular.

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Total elapsed time = 62575s. Finished at l = 64, y = 453059
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The Tribulation (continued)

12/21/06 9 P.M. Started mod 256 computation on sage. Computed 452,174 out of 453,060 rows of KL polynomials in 14 hours, then sage crashed.

12/22/06 EVENING Restarted mod 256. Finished in just 11 hours

(hip, hip, HURRAH! pthread_join(cheer[k], NULL);)

Total elapsed time = 40229s. Finished at l = 64, y = 453059
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VmData: 54995416 kB

12/23/06 Started mod 255 computation on sage, which crashed.

sage down til 12/26/06
(regional holiday in Seattle).
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VmData:  54995416 kB

12/23/06  Started mod 255 computation on sage, which crashed.

sage down till 12/26/06
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The Tribulation (continued)

12/21/06 9 P.M. Started mod 256 computation on sage. Computed 452,174 out of 453,060 rows of KL polynomials in 14 hours, then sage crashed.

12/22/06 EVENING Restarted mod 256. Finished in just 11 hours

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consult experts ↩ probably not Sasquatch.

Did I mention sage is in Seattle?

Decided not to abuse sage further for a year.

1/3/07  Atlas members one year older ↩ thirty years wiser as team ↩ safe to go back to work.

Wrote KL polynomials mod 253 (12 hrs).

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Chinese Remainder Theorem (CRT) gives answer mod $253 \cdot 255 \cdot 256 = 16,515,840$.

One little computation for each of 13 billion coefficients.
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Marc van Leeuwen started his CRT software. On-screen counter displayed polynomial number: 0,1,2,3,...,1181642978. Turns out to be a bad idea.

Restarted CRT computation, with counter 0,4096,8192,12288,16536,...,1181642752,1181642978. Worked fine until sage crashed. William Stein (our hero!) replaced hard drive with one with backups of our 100G of files mod 253, 255, 256.

Re-re-started CRT computation.

Output file 7G too big: BUG in output routine.

Marc found output bug. Occurred only after polynomial 858,993,459; had tested to 100 million.

Re-re-re-started CRT computation.
The Chinese Remainder

1/4/07 **Marc van Leeuwen** started his CRT software. On-screen counter displayed polynomial number: 0,1,2,3,...,1181642978. Turns out to be a bad idea.

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1/5/07 **AFTERNOON** Re-restarted CRT computation.

1/6/07 7 A.M. Output file 7G too big: **BUG** in output routine.

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So what was the point?

In the fall of 2004, Fokko du Cloux was at MIT, rooming with fellow Atlas member Dan Ciubotaru. Fokko was halfway through writing the software I’ve talked about: the point at which neither the end of the tunnel nor the beginning is visible any longer.
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Fokko was startled by this remark, but not at a loss for words. “I don’t know about you, but I’m having the time of my life!”
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Fokko du Cloux  
December 20, 1954–November 10, 2006